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On the Foundations of the Euclidian Geometry

Definition 1) Geometry is the Science which treats of spacial forms and magnitudes and their mutual relations. Dealing with magnitudes and with spacial forms only in so far as these are determined by their magnitudinal relations Geometry is a division of the general Science of quantity – Mathematics.

A few introductory remarks are necessary in the way of more accurately specifying the object of geometry which I prefer to put in the form of definitions. These definitions, however, will not be only nominal, most of them prove the actual existence of the objects they define

Definition 2) Space is that in which all bodies exist. It is the condition sine qua non of material or extended objects. This truth is expressed in physics by the assertion that matter or the substance of which all bodies consist, has extension or, in other words, material objects occupy space.

Definition 3) Experience teaches us that matter is also impenetrable, i.e. that every material object occupies a definite portion of space which is fixed by certain limits or boundaries and which cannot at the same time be occupied by any other material object. The portion of space that is for a time exclusively occupied by a certain material object is called the place of that

object for the sake of accurate terminology I propose to call it the geometrical place.

Definition 4) Experience however teaches us that the resources of space with regard to its capacity of containing material objects or of affording place to the material substance are absolutely limitless. Thus notwithstanding the impenetrability of material substance, explained above beside any occupied space there is always room enough for the existence of other material objects and any vacant space is always conceived as susceptible of being filled up with matter. Moreover, any material object is conceived of as capable of being divided into any number of portions and these again subdivided into lesser ones, and so on, ad infinitum. In this respect extended substance and hence, also space follow perfectly the nature of abstract quantity, ranging both ways from the finite to the indefinitely small on the one hand, and the indefinitely large on the other.

Definition 5) The geometrical place of a body, being that portion of space which is occupied by the body to the exclusion of any other body has the same spacial form and dimensions as the body which fills it up. We mean by this that whatever measurements in regard to extension the whole body, or its several parts, may have, the same are attributable to its geometrical place and whatever arrangement of extended parts makes up the form of the body, the same belongs likewise to the geometrical place, and vice versa - what is true

respect the geometrical place may be substituted for the body, and conversely. This geometrical place alone apart from all other physical properties or, in fact, apart from the matter filling it up is dealt with in geometry, - and is regarded by this science 1) as a magnitude - that is, not only as something that can be greater or less, the

reason for which is given in Definition (2). 1) A magnitude is anything that can be measured, that is accurately compared with a standard magnitude of the same kind which is arbitrarily taken as a unit and can be repeated any number of times or divided into a certain number of equal parts thereby becoming either equal or greater and smaller than the magnitude in question. and 2) as a form, consisting of a definite arrangement of parts according to some law which can also be expressed by numbers.

Definition 6) The geometrical place of a body is called a solid in geometry meaning by it that it is mentally represented as preserving a fixed form and dimensions. The geometrical solid is a mere ideal abstraction and has nothing to do with physical solidity from which, however, it is originally derived. Geometry does not, however, treat it as impenetrable. It is indeed only the impression left by a body in surrounding space, conceived of as capable of preserving the impression after the body itself has been removed. As a magnitude or thing to be measured and expressed in numbers, without regard to its form or outer appearance it is called a volume.

Postulate. 1) The geometrical solid or body may be, mentally imagined as moving about in fixed space, or changing its position w. respect to other bodies, whether physical or geometrical, without

distortion or change of form. The solid is said to possess geometrical rigidity, meaning by it that the disposition of the parts with respect to each other is fixed and unchangeable or that there is no internal motion. This idea of geometrical rigidity is derived from the fact that space is conceived of as affording a mere passive capacity of being filled up with matter, all changes of form being referred to the active principle of the material substance proper, i.e. physical causes alone (cf. Ref. 4). It is also based upon the undoubted fact of universal experience that, in so far as can be ascertained by observation and experiment (measurements - astronomical physical and geodesic), no real or physical rigid body moving about in space has ever been known to undergo any alteration in form or dimensions on account of change of position in space alone, without regard to physical causes which, in most cases, have been found quite adequate to account for such alterations. As a matter if the contrary were the truth in the case of real bodies the Euclidian geometry would still have nothing to do with such alterations, as it considers only ideal rigidity where change of form or dimensions as depending upon position is purposely eliminated for the sake of simplicity, and may be left to other branches of the mathematical sciences to consider. Kinematics for instance, may well properly consider such questions as a special

kind of lines constant, [illegible] meters those of position included). But the Euclidian geometry considers only the simplest case, even if it were only an idealized abstraction.

Definition 7) A body is said to be equal to another geometrically when their geometrical places can be made to fill one another without remainder of any parts of the one not filled by corresponding parts of the other. When the geometrical places thus fill one another, we say that the geometrical bodies coincide – coincidence as thus defined, being a proof of equality invariably resorted to in geometry. When the coincidence can take place only with the change of form resulting from a mere rearrangement of parts, that is, preserving separately their respective magnitudes and forms, the bodies are said to be of equal volume though not equal in form

Definition 8) One body is said to be greater than another when some of the parts of the one can be made to coincide with all the parts of the other, while there still remain some parts of the first, having no corresponding parts in the second to coincide with. The other body is then called the less.

Scholium. Having firmly established the empirical and rational basis of the notions contained in the previous definitions,

it may not be amiss to give a more compact and abstract form to the logical process by which they are obtained, which is free from all cavils from those who think that spacial forms and magnitudes may for all we know be certain functions of absolute position, which we shall never be able to ascertain or disprove. Starting with the notions of space matter or extended substance in general, position and change of position or motion and with the abstract notion of quantity, we may assume for the sake of abstraction, the existence of an hypothetical impenetrable material substance infinitely divisible, i. e. possessed of the following properties:

1st Impenetrability.— Every determinate portion or quantity of this substance occupies or fills up a corresponding portion of space which cannot at the same time be occupied by any other portion of the same substance. Any two portions of space thus filled up by the same quantity of the substance at different times are said to be equal in capacity, and any two portions of the substance which can fill up the same portion of space at different times, or different portions of space of equal capacity at the same time, are said to be of the

same bulk. So that to each bulk which measures the quantity of the hypothetical substance there is a corresponding capacity of the space which is occupied by it at any moment to the exclusion of any other portion of the same substance; to a greater bulk of the substance there corresponds a greater capacity of the space occupied by it, to a double or multiple bulk, a double or equimultiple of the capacity of the space, and to any part of a given bulk a corresponding part of the capacity of the space occupied. The generic term for bulk or capacity alike is volume, so that the quantity of the substance (bulk) and the space filled up by it to the exclusion of any more of the same substance (capacity) are said to be of equal volume.

2d Infinite divisibility. — If a portion of the substance is divided into n portions such that they can fill up spaces of equal capacity each, that portion is said to be divided into n equal parts. The property of infinite divisibility now becomes perfectly comprehensible and is possessed according to hypothesis, both by the hypothetical substance and by the space giving position to it, substance.

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form; rigidity, or plasticity. — If two equal portions of the substance (of equal bulk or volume) are made to fill up successively the same fixed portion of space (not merely spaces of equal capacity) then in the two portions of the substance we observe not only equality in bulk of the whole, but also of corresponding parts, filling up in the two cases the same corresponding parts of the space; that is the two portions of the substance, each while filling up in turn the space considered have a similar arrangement of ~~equal parts~~ unequal numbers ~~each~~ parts which are equal in bulk in the two cases, no matter in what manner the division is made, and however small the parts considered. The two equal portions of the substance in their successive positions, are, therefore, said to be not only equal in volume, but also in form, — equal form thus meaning an equal arrangement of equal parts. When a portion of the substance leaves a certain position, passing into another position, it may change its form, i. e. the arrangement of parts may change so that an arrangement, denoted by a, b, c, ... k, may now have to be represented by ... c ... k ... f ... g ... a — b ... Moreover, even if it does not change its form, that is in some one part in it at least, reverses its old position the form of the whole may still change, ...

where a, b, c, ... f, g, ... k do denote unequal numbers.

actually does change whenever the remaining portions, their relative positions to this [illegible] and to each other - and provided the space filled up by the whole is still continuous, that is it is filled up compactly and represents one concrete whole without interruptions of vacant or unoccupied portions intervening, we say the space occupied by the whole has not changed its volume, but has changed its form, and so has the substance filling it up. The distinction between two portions of space of equal volume and different form and two portions of space of equal volume and equal form is now clear and unambiguous. Indeed, we have seen that when the substance as a whole does not leave its original position, i.e. when at least one of its portions preserves its position, a change in form is possible only in virtue of the change of position of the remaining parts w.r. to the stationary part and w.r. to each other; in other words, change of form is caused by motion of parts of the whole w.r. to each other, that is by internal motion. The same is, therefore, true w.r. to a substance which has left its original position entirely, filling now up a portion of space which has not the smallest part in common with the original position. If there has been internal motion or a rearrangement of unequal parts besides, in the substance has also changed its form;

if there has been no internal motion, the original arrangement of parts having been preserved, the substance has only changed its position, but not its form. If a substance resists a change of form, as just defined, whether the whole is at rest or in motion, we may say that the parts are held fixed towards each other and we call this state of the substance the rigid state; the whole substance is then said to form a solid or a rigid body. The portions of space which represent any two successive positions of a solid in motion are said to be equal to one another in volume and form, just as two solids that can be made to fill up successively the same space are themselves said to be equal in volume and form. When two equal solids are made successively to fill up the same space, then by abstracting from time, that is, disregarding the fact that the filling up can take place only in different moments, we simply say that the two solids are made to fill up the same space or they are made to coincide with one another, coincidence being a test of equality in volume and form. If, on the contrary, the substance does not resist a rearrangement of parts, these parts are not held rigidly towards one another, and change of form is possible without change of

a portion of the substance is said to possess plasticity.

We see now that these notions though having a firm empirical basis, are not absolutely dependent upon the actual condition of things. The hypothetical substance, of absolute impenetrability, need not actually exist, but as an abstraction, agreeing, in general, with our experience, it may serve as a starting point for the only possible science of measurement of extension, since the notions based on its assumption are clear and unequivocal, and absolutely necessary to make the investigation of the laws of spacial forms and magnitudes possible. Moreover, we must agree to class all actual phenomena, in so far as they conform to the laws deducible from these notions and from those that follow in this introduction, as geometrical phenomena, that is such as depend upon the essence of extension only; and in so far as they deviate from these laws, they must be explained by physical causes, - and any attempt to confuse the two (as very able geometers, like Clifford and others have done) would only tend to raise a dust of endless discussion which would never permit us to see the real foundations of geometry.

finition 9) A rigid physical body is said to be surrounded by
meant space on all sides, when it can be moved in
all directions: forwards and backwards, to the right or
left, up and down, and so on, in all intermediate directions[1]
When two rigid physical bodies are such that
in any part of it, the two bodies are said to be at a distance
from one another that admits, either the position of some
other body between them, or the motion of one towards the
other. In the last case the bodies are said to approach
one another, the distance between them becoming less
and less, until it vanishes altogether admitting of
no further approach towards one another, the bodies
then being in contact. These ideas of distance and con-
tact are transferred upon geometrical solids or the
geometrical places of bodies corresponding to the
positions of physical bodies, just mentioned.

efinition 10) When two rigid physical bodies are brought
into close contact with one another, so that no further
motion of one towards the other is possible, they are said

(7)

1) The word direction is used here in the common acceptance
of its meaning, viz. some course, but it is really vague. The
scientific meaning of this word will be given in another
place in this work.

to have reached the limits or boundaries of one another; and if these limits are to some extent continuous, i.e. when they touch one another in many parts, the touching [illegible] uninterrupted by intermediate vacant space, which happens when the bodies fit to one another, the limits are then called surfaces. A physical surface is, accordingly, the continuous boundary where the rigidity of a body just begins or where the physical property of impenetrability just begins to act. If the body is surrounded by vacant space, the surface of the body is the boundary separating the impenetrable matter of the body from the capacious space. But it is neither the one nor the other since the smallest part of the body has some of its parts removed from the boundary by the interposed rigidity of other parts of itself. No other rigid body could possibly have access to those concealed parts without overcoming rigidity, and, therefore, no part of the body however small, can belong to its surface, of which the essential character istic ~~previously~~ is its being in contact with some other body, or with vacant space and, hence, being capable of being brought into contact with some other body. Surface therefore, has no magnitude of the same kind as a body in other words, it has no bulk or volume, and it can never amount to

any part of volume. But as is shown in the following definition it nevertheless has magnitude and form of its own; in other words, it is a thing that can be measured and expressed in numbers, these numbers being in determinate relations to those expressing the volume of the body bounded by the surface. One of the tasks of geometry is, in fact, the discovery and determination of these relations. The idea of surface is also transferred from physical bodies to geometrical solids, and the geometrical surface may be said to represent the geometrical place of a physical surface. Geometry considers it as a separate entity, capable of existing by itself and moving about in space or changing its position w. r. to other bodies and surfaces, whether physical or geometrical, without distortion or change of its form.

Definition 11. Since a rigid body, immersed in occupied space, or in any plastic material substance, displaces a portion of this material or occupies a portion of the void, to the exclusion of other matter, equivalent to its own volume, and since this rigid body exposes only its surface, the interior

parts not coming into play at all in the act of this displacement or exclusion (the interior might as well be imagined hollow or devoid of matter in this connection), it is evident that, in general, surface ought to be a function of volume, increasing with the increase of the last, that is, to a large volume there must in general correspond a large surf.; although the converse is not a necessary consequence. At any rate, it is quite inconceivable how a rigid impenetrable body could take up space to the exclusion of other material substance which, on account of its capability of motion or of change of position, can be prevented from occupying the same space as the body considered only by the boundaries of the last, were it not that these boundaries are in themselves an extended magnitude, standing in some functional relation to volume and form. Accordingly surface must have portions, all of which may be exposed to vacant space, or in contact with surfaces of other bodies, or some exposed and the others covered by corresponding surfaces of other bodies. In this last case the exposed surfaces can again be brought in contact with surfaces of other bodies. Two surfaces will then be equal if they can lie upon one another and mutually cover all its parts.

and one is greater than the other when a part of the first can cover the whole of the second, while another part of the first will remain exposed or covered by a third surface.

Petholium. Any part of a body may be regarded as a separate body, in the geometrical sense of the term, from the remaining part, since each can be imagined to move about in space independently of the other, and without distortion or change of magnitude, - and while the two constitute the parts of the same solid, the limit common to both is a surface of some definite shape and magnitude.

Corollary. Surfaces coincide with each other, when the bodies limited by them coincide and, conversely, bodies limited by surfaces that can be made to coincide with each other, must themselves be capable of coincidence, since when the surfaces are brought into actual coincidence, none of the bodies can help being everywhere within and nowhere beyond their coinciding limits

Definition 12. Both experience and reasoning lead us to the conclusion that while volume, or unspecified space, space in all possible directions, wherever motion is possible, is homogeneous, — surface or that which limits a body may be of very different kinds,

having almost nothing in common, except that an indefinitely small or infinitesimal part of one surface may be imagined to move towards another infinitesimal part of the same surface or a continuous infinity of infinitesimal parts in the surface itself, as will be shown. But this common property is not sufficient to make surface a magnitude always definable with mathematical precision and capable of being expressed in a voluntarily chosen unit. An indefinite size of the small part that is capable of congruence would make the computation of surface area with mathematical precision impossible unless there be a way of reducing these to surfaces capable of coincidence in finite parts. Volume as a magnitude is, in fact, only the capacity of space to contain matter of a constant ideal impenetrability, is everywhere the same. Any part of volume is capable of coincidence with any other corresponding part of volume; this conception is given directly in the fundamental idea of motion, together with the idea of rigidity of the moving bodies. Space, therefore, or space unspecified is homogeneous; whereas surface as having an infinite variety of forms, is not so. And while, for instance, a smaller body being posited within or surrounded by a larger one, it corresponds a part of the surface of the larger, the limits of two bodies may be and, in fact, may be unequal

[illegible] incapable of coincidence in any of their finite parts. In order to make surface a mathematical magnitude i.e. defined with precision there must be found at least one homogeneous surface of which any finite part is capable of coincidence with any other corresponding part of the same — and taking such a surface as a standard we must be found able to refer their surfaces with any desirable degree of precision to this standard. Such homogeneous surfaces, the essential characteristic of which is that any part of them can be imagined to slide upon the whole remaining always in coincidence in all its parts with corresponding portions of the whole do really exist, and their existence is likewise both a matter of experience and mathematical deduction. Any of these homogeneous surfaces might be taken as a standard of measurement, but one affording the greatest advantages for computation and being capable of indefinite extension, is accepted as the standard and all others are always reduced to the single standard. It is needless to remark that a difference of choice of the standard surface would, like different systems of numeration, only lead to different methods of computation, but not to different results.)

To sum up: Surfaces are multiform and are, therefore, seldom capable of congruence in their finite portions. The nature of measurement, however, requires a homogeneous standard to which all magnitudes of the same kind that are to be measured can easily be reduced. Surfaces answering this requirement of homogeneity and hence capable of serving as standard of measurement of area and form actually exist, of different species and infinite in number the essential characteristic of all of which is the capability of any portion of such a surface to slide along the side, remaining always in coincidence with different corresponding portions of the whole. The simplest of these is chosen as the norm; it will be shown that it possesses the additional properties of being unlimited extended ~~on all sides~~ beyond any arbitrary limit and its side, towards the interior of the body which is limited by it, fitting upon the opposite exposed side, so that the two can be made to coincide (plane).

(cf. fig. 13). If two bodies are in partial contact of their surfaces with one another, the boundary separating the part of surface in contact from the part not in contact, in either of the bodies, is the limit of either the covered or the exposed portion of the surface [illegible] let us imagine the surfaces to be homogeneous. Any finite portion of the uncovered surface can be imagined to be in contact with the surface [illegible] (See fig. 11), whose form at a finite distance is immaterial and which moves upon

the rest of the uncovered surface, along any path in it, until it reaches the limit of the covered surf., where it is [illegible] by the rigidity of the surf. of the covering body, and can only move so that while a portion of its surface touches the covered body, another portion touches the covering body. Motion along the boundary, separating the covered from the uncovered surface, must still be possible for an indefinitely small portion of a finite body, whose contact with the two bodies in the exposed portions of their resp. surfaces, blends along the boundary for the infinitesimal element; for, as this finite body can be conceived to move along the two, remaining always in partial contact with either, the infinitesimal element, touching the two simultaneously, must of necessity find a region of motion along their common boundary. The boundary will, therefore, have parts of its own, viz., the specializations of position of the infinitesimal element touching; considered: but these parts will not be of the same kind as the parts of a surface, neither can the whole be a part of the surface. In fact, it cannot be a part of the covered surf., since it must likewise belong to the uncovered surf., but however small a portion of the covered surf., if not infinitesimal, will always [illegible] some of its parts [illegible] from the exposed region by intervening parts of the covered region. Similarly, it cannot be a part of the exposed surf. But we have proved that it must have parts of its own. The boundary between two portions of surf. is, therefore, a new magnitude; it is a line, and its parts are different from those

of a solid or a surf., but like them expressible in numbers having some determinate relation to the numbers expressing the magnitudes of volume and surface. ⊕

eholium The analysis of last definition can be made more concrete by the following [illegible] which also puts its result in a somewhat different light:

When three bodies touch each other in their surfaces and they also fit to each other, so that no vacant space is left between the parts of the touchings, the same part of surf. can belong only to two of the touching bodies if [illegible] boundary separating the surf. belonging to the pair simultaneously, from a surf. belonging in common to either of the pair and the remaining third body, belongs to all three bodies simultaneously and hence, is not a surf., but a line. It is the continuous boundary

⊕ The same reasoning holds also in the case of non-homogeneous surfaces, provided a certain amount of plasticity of form is allowed to the touching parts of the moving body.

of no entity of three bodies that have come into contact with each other and may indefinitely small part of it can be conceived to ~~any other~~ pass to the position of any other of the same body by two different courses in case the boundary is ~~closed~~ connex and the line returns into itself and only by one course if the boundary is not completed. +

The locus of line is distinct, relation of physical surf upon a geometrical one, and the geometrical line may be said to represent the geometrical place of a physical line. Geometry regards it as a separate entity which can be conceived to move about in space (or of any invariable surf.) without change of form or magnitude.

Defin. 2) That in which the statement made in Defin. 1) that an infinitesimal part of the same surf. by a continuous infinity of diff. cont. paths can be moved towards another infinitesimal part of the same surf. by a infinitesimal part of the same surf. itself.

Defin. 14) Any body may be regarded as divided in two definite parts having one part of their surfaces, namely, that covered by the section in mutual contact, while the surface of the original whole body is now also cut in two parts, each belonging to one of the two bodies, now taking the place of the original one. The boundary, separating the common surface from the distinct parts, will, according to Defin 13), be a line on the surf. of the original body, thus having from one side, at least, exposed to vacant space. Any part of this line may be brought into contact with some other body. When the whole line is in contact

with other bodies leaving no part of it adjacent to vacant space, the whole line can be regarded as covered by another the duplicate of the first in form and magnitude which is traced upon the surfaces of the touching bodies, and no body besides can be in contact with the line. The line and its duplicate are then said to coincide, coincidence in this case also being a proof of equality in both form and magnitude. If we form in a similar way the duplicate of only a portion of the line, this duplicate will evidently be less in magnitude than the whole line.

Cor. Lines coincide with each other when the surfaces limited by them coincide, since none of the lines limiting these surfaces can be without or without the limited surfaces. The converse, however, is not necessarily true, with the exception of ~~homogeneous surfaces~~ (except in the case of the plane, as will be shown later).

Schol. Lines like surfaces are multiform but there exist also homogeneous lines, which are such as is capable of coinciding with every one of its parts or the same. One of such homogeneous lines, capable of infinite extension and of being determined by any two of its elements, is accepted as the standard of line measurement etc. [illegible]

Defin. 5) When the surfaces of two bodies are in partial contact, the boundary line being, in its turn,

brought in partial contact with a corresponding line upon the surf. of a third body, or in other words, a part of the line being covered, an indefinitely small part of the remaining uncovered line may again be covered with a corresponding infinitesimal line upon the surface of still another body, and the parts of the last body immediately adjacent to the line in question imagined to move along the uncovered part of the line, only by two opposite paths, until they reach the limit of the covered part of the line, where further motion is checked altogether by the rigidity of the surface of the body which has effected the first partial covering of the line. This limit, therefore, separating the covered part of the line from the uncovered, has no parts at all, since an infinitesimal element coming up to it finds no extension to move upon (there would be no advantage in this case to start with a finite part of the line moving upon the uncovered part by allowing plasticity of form in case of non-homogeneity since a line being a region of motion only for an infinitesimal part of a time touching both surfaces at once, the motion upon it must be a filing in or successional motion all along the line that is the motion of a row of individual members where there is a uniform succession of each member into the place of one immediately preceding going either indefinitely or returning to the original starting place the members being the infinitesimal portions of the line and all

regarded in geometry as a separate entity, which can move about in space independently. It is neither homogeneous nor heterogeneous, since it has no parts. The geometrical pieces of two pts. always coincide with one another as soon as they are brought in space, within a body, upon a surf. or along a line into the same position. It is, therefore, regarded as the element of space

Cor. When two lines coincide, their ends or the pts. limiting them coincide also, i.e. these ends have the same positions, two by two

To sum up what has been stated in the previous definitions: The boundary separating impenetrable substance from capacious space, or the region where no motion is possible from the region where motion is wholly unimpeded, is a surface and admits only motion in contact along finite regions with the condition of plasticity of the touching surface of the moving body for the case of non-homogeneity.

The boundary separating uncovered surf. from covered surf. or the region where motion in contact is possible from the region where such motion in contact, for finite regions, is also impossible is a line and being with the first nor the second, it admits motion in contact for an indefinitely small portion of a body.

The limit separating an uncovered portion of a line from a covered portion, or that region where motion in contact is possible for an infinitesimal portion of a body from the region of absolute exclusion of motion, is a point, which is, thus, the position of an infinitely small portion of a body at rest; it has, therefore, no dimensions and only position.

Space in its totality being the repository of extended substance, which is capable of motion (change of position) and endowed with the properties of impenetrability, rigidity and infinite divisibility limiting and bounding vacant space in certain definite ways, gives rise to three different kinds of spacial magnitude so connected that one is the limit of the other and is itself the limit of a third. The point is the result of limitless divisibility of any of the three kinds of extended spacial magnitude.

Space is, therefore, a tri-dimensional manifoldness only because of its three chief attributes giving three different kinds of specializations of position limiting each other and so connected that there is always a certain determinate relation between the units of [illegible] of space and the units of the other kinds as is shown in defin. II) that the rigid surf. by means of which the property of impenetrability makes itself effective must be some

function of the volume; reasons were also given why surf. should be a field of motion for finite regions in contact, and a line a field of motion for infinitesimal regions of contact from which would follow at once that a line is a differential element of a surface and a pt. an element of a line. Unlimited space in its totality would be a one dimensional magnitude, ranging from zero to infinity, in terms of volume alone, were it not for the invariable relations between the units of volume and those of the two subcategories of specified space, resulting from the fact that these always limit each other. In order, therefore, to suppose that space has more than three dimensions, we must conceive that a pt. has dimensions, since we began our analysis from free, unlimited space, and found it in itself, without considering its limits, to be one-dimensional; and only in relation to its limits it becomes tri-dimensional, since its first two limits (but not the third) are also magnitudes and bear certain fixed relations to unspecified space as a magnitude. Since u, du, d^2u, where u represents a piece of unspecified space, i.e. volume, are all variables, i.e.

follows that if u is a fnct. of x, it must be of the 3rd order: $u = ax^3$ may do for the simplest representation of such a fct, where dx will represent the differential of a line; it may of course, and generally it will be a homogeneous fct. of the first degree of a number of differentials $dx_1, dx_2, dx_3 \ldots dx_n$ but then there must be $n-3$ linear relations bet. the dx's, reducing the number of independent ones to 3.

It is absurd by reasoning in a reversed order to infer by a kind of induction that just as a pt. in moving generates a line, a line in moving out of its region generates a surf. and a surf. generates a body, so tri-dimensional Space in moving out of itself will produce a new kind of space. In the first place we could have to prove in general if our reasoning holds for n if for $n+1$ as we always do in mathematics with such a kind of induction. Riemann's construction of an n-ply variability out of an n-fold and a variability of one dimension, is based on the assumption that the n-ply variability passes over into another one, entirely different, in a determinate way, so that each pt. of the first passes over into

a definite pt. of [illegible] other which is not at the same time a pt. of the first — an assumption that must be proved in every particular case. In our case this assumption is actually equivalent to the assumption that space can move out of space, which is absurd by the very definition of space, 143. Space is that which gives place to material objects, whether at rest or in motion; so that wherever motion is at all possible for a tridimensional piece of space, there is space again. In the second place, even if we admit the possibility of space moving out of itself into some other region we have not admitted any new property of space which might be the object of measurement since we have only assumed that the new region is not space, so that space would remain again tridimensional. And even if we should admit a new unknown property of space, we would still have to prove, that tridimensional space is its limit and that its units bear a fixed mathematical relation to the units of space we have already considered. Indeed, it is not sufficient that a phenomenon have a certain number of properties in

continuation of p. 2

order to consider that phenomenon of as many dimensions as the number of its properties, the properties must be the limits of comparison and their units must stand in a certain invariable mathematical relation to each other.

Definition 1). A pair of fixed points in space or any two points in rigid connection which can be made to coincide with these are said to be at an invariable distance from one another.

Cor. 1) If a pair of points A and B in rigid connection are capable of coincidence one by one with another pair C and D, likewise in rigid connection between themselves the two pairs will be at equal distance from one another i.e. distance of A from B = distance of C from D or of D from C.

II) Two pairs of fixed points in space A, B and C D both of which are capable of coincidence successively with the same freely movable pair in rigid connection E, F, are at equal distances from one another.

For shortness we shall call a pair of points in rigid connection, free to move as a whole, simply a rigid pair of points and will denote them thus (AB).

Definition 1) If any surface or line can be made to coincide with another surface or line, it can do so only by passing from one position to the other by a continuous path, every position in which is a surface or line resp. congruent with the moving one

Lemma) From the principles of rigidity and the definitions of body, surface and point, it follows that a body is absolutely fixed in space when and only when a finite portion of the surface limiting it, or in any way rigidly fixed in it is fixed in space. Hence a body

and, therefore, also any surface or line, rigidly fixed in it, may be conceived to move when only one point in the body, the surface or the line is held fixed in space. Such a motion of the body is called rotation, revolution or turning about the fixed point.

(Theorem 1). If (AB) denote a rigid pair of points of which A is fixed and (B) is made to assume all possible positions compatible with the rigidity of the pair and the unity of A, then (B) will describe a homogeneous surface called a spherical surface limiting a body called a sphere. A is called the centre of the sphere or the spherical surface and the invariable distance AB, from the centre to any point in the surface, is called the radius of the sphere.

In fact, the moving point (B) can pass from any fixed point B with which it initially coincided to any other point B', B'' and so on by a continuous infinity of paths crossing and recrossing each other in all conceivable ways provided all these points are at the same distance from A as B. Let (B) pass from B to B' by a continuous path of some determinate rigid form BB'; while doing so any rigid line connecting A and (B) will describe a portion of a surface. This surface in its turn, conceived as a rigid form, can be imagined to move while A is fixed (according to Lemma) so that every one of its points describe a line not already contained in the original position of the surface itself, thus describing a body. The line (BB') conceived rigid and always limiting the moving surface (ABB'), will then be dragged along with it in its motion and will describe

(Preceding Lemma and Theorem 1).

a surface since each of its points describes a line, not coincident with the original position of the line itself: in other words, each point of the moving line (BB') passes to another point, not already contained in the original position of the line. Moreover, since the surface (ABB') can be conceived to sweep through all that portion of space around A, whose aggregate of points is characterized by the property that their distances from A are correspondingly equal to those of the aggregate of points contained in (ABB') in its original position, it follows that the surface described by the line (BB') will contain the whole aggregate of points which are at the same distances from A, as (B) in its original position. It is also evident from the mode the body described by the surface (ABB') is generated, that it is a continuous body, i.e. the aggregate of points composing it is a continuous aggregate, which allows to pass from any point in the body to any other through any third, belonging to the body by a continuous path lying wholly in the body (i.e. every point of which belongs to the body). The surface, therefore, is also continuous in the same sense namely, that we can pass by continuous motion from any point in it to any other through any third, by a path every point ~~every point~~ of which is in the surface. Moreover, the moving point will during such a motion remain at the given distance from A, hence, on the surface of an imagined fixed sphere, - and if the moving point is conceived to be fixed in

surface of a sphere of same centre A but which is dragged along with (B) in its motion we see that every portion of a spherical surface is congruent with every other of same limits, that is to say the spherical surface is a homogeneous surface.

Cor. 1) Since the spherical surface separates a continuous portion of space from all other space, that is, all points that can be reached by one continuous path wholly contained in the body, from all points of space that can not be reached by a continuous motion from the centre unless the boundary of the body is crossed by the path of which a portion at least does not belong to the body - it follows that this surface is also a closed surface.

Definition II) All points in the sphere (body limited by the spherical surface) generated by the given radius AB, from the given centre A which can be reached by continuous motion from the centre without crossing the surface at all, or after crossing and recrossing it an even number of times, are said to be within the surface, and the distances of all these points from the centre (excluding those of the surface itself) are said to be smaller than the distance AB. All points that can be reached by continuous motion from the centre only after crossing the surface once, or crossing and recrossing it an odd number of times, are said to be without the surface, and their distances from A are said to be greater than the distance AB. Such points are also said to be without the sphere.

Cor. II). When a sphere is conceived to move within the boundaries of a fixed spherical surface, which is always in coincidence with the surface of the moving sphere, its centre (or centres if there can be more than one), remaining at a constant distance (or constant distances) from each and every one of the same fixed aggregate of points belonging to the same surface, must remain fixed in position. But since when the sphere moves around one fixed centre, from which it is conceived to be generated, every other point in it, at a distance from the centre, moves upon the surface of a sphere (Theorem 1), it follows that given a spherical surface as a whole, its centre is uniquely determined. A complete spherical surface is therefore, said to be the locus of all points equidistant from a unique fixed point, called the centre.

Cor. III). If two spheres coincide in any finite portion of their surfaces they coincide throughout and, hence, have the same centre and radius.

For in holding the co[illegible] finite portion of each spherical surface fixed, each of the surfaces remains fixed. (Lemma 1) But if either of the spheres is conceived as moving within its fixed boundaries around its fixed centre every other point, at a distance from the fixed centre, moves upon a corresponding sphere and passes into the position of every possible point on the last and of no other point. Hence there is only one centre common to both since the portion of each movable spherical surface which is initially in coincidence with the corresponding common portion of the two fixed surfaces, can without leaving either of the spherical surfaces, pass over into any congruent portion of either.

[illegible]

It follows from what precedes that a dist. has is a geometrical magnitude. To two different pts. from a given pt. distances can be connected with the same pt. by the same line which might serve as a path of motion from one end pt. of each dist. to the other. If we take, to fix ideas, as the connection bet. A and B — a line of some determinate shape, lying wholly within the sphere described by AB, a concentric sphere described by a distance AC less than AB, will cut every line (AB) into two parts, one of which will lie within the smaller sphere and the other, outside. Thus we see, that to a smaller dist. corresponds a smaller path of motion from one end of the dist. to the other. This suggests the idea of representing the distance magnitude by a line. But then we must find a line such that to a fixed dist. from one fixed pt. on it to another should correspond a unique portion of the line and to a smaller portion of the line should also correspond a smaller distance which is not always the case with any line. Is there such a path between A and B? I say that if we eliminate all paths between A and B such that can assume different positions while A & B remain fixed since their dist. is unaltered, there must still be left a path or [illegible] them which is unique for

~~for~~ this fixed position of the ends and which will also satisfy the other requirement, as I shall proceed to prove with the utmost rigor in the following two proposition and their corollaries.

Measurement of distances from a fixed origin, addition and subtraction of a given distance.

Theorem 2) Let S represent a sphere ~~surf~~ described by radius a from origin O. OA=a, as soon as A is on its surface. Describe from A a sphere S' with radius b, where b>a. Then O is outside S' (Cor. 2), and hence some portions of S are outside S', for if the whole of S were inside of S', the center of S would be there a fortiori. Similarly some portions of S' are outside of S. But some portions of S are also inside S', for ~~otherwise, it would be impossible to reach by continuous motion from S separates A and a portion of space around A~~ any point belonging to S, before crossing S' once, which ~~all exterior space~~ is absurd, since A itself is on the surface of S, and hence ~~can not have access to the exterior space without~~ belongs to S ~~crossing S, now if some portions of S contiguous to A are not inside S', these portions and also of S will be outside S', which is absurd~~; hence some portions of S is inside S'. Also some portions of S' are inside S, for if S' were wholly outside S, S containing ~~only its interior points, would be~~ ~~be either wholly interior or~~ wholly exterior to S', ~~which~~ which hypothesis has just been proved impossible. Hence the two spheres ~~and their surfaces~~ penetrate ~~each other~~ having a portion of space in common, limited by the portions of their surfaces which they have inside one another, and two other portions of space,

Le Roux: This remark belongs to page 21, bottom.

† By continuously diminishing the radii of the circumscribing sphere and dropping the perpendicular on its plane thro. a pt. within S' we shall at last arrive at a series of spheres which have all their volume outside S', and have not one pt. in common with the constant sphere. It is evident, therefore, that in this process of passing continuously from spheres having some portions inside S', to such whose volume and surface are wholly exterior to S') so that there is an appreciable distance from which and every pt. belonging to the region of these exterior spheres and every pt. belonging to the region of S' — we must on the way encounter a sphere such that any sph. described by a rad. greater than its own has some portion of its vol. inside S' and any sph. described by a smaller radius has every pt. of its region at some appreciable distance from the region of S'. And, in crossing from the series of greater to the series of smaller spheres we obviously can choose two, each from either series, whose radii differ as little as we please from one another and from the radius of the bounding sphere. Evidently, the minim. sphere of the first series and the max. of the second series both tend towards the same line, — the bounding sphere, which, we say, touches the constant sphere. This sph. must have at least one pt. in common with the constant sphere, such many have more pts. over a continuous line of contact until of course it becomes, which is done in a flash. Similar remarks hold in the case of the present demonstration, with which we shall go on, instead of substituting [illegible]

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Now they cannot touch in several "dis
for any line of fixed form connect
with C and it could be displaced so

A
O

A
O

remain fixed, the pt. on the line, which has been in coincidence with the touching pt. originally taken, shall coincide with any other pt. in the common touching part. which must be at the same distances from A, &c.; &c., since by the principle of continuous congruence in motion (Axiom I), the fig. described by the displaced line is a continuous one, & the fig. described by a determinate pt. in it is also a continuous one. It must, therefore, be either one line, any portion of which can, by a continuous motion along itself, pass, without deformation, into any other of some finite, or one, pt.

Now it cannot be a line, for this would imply that the surf. of the sphere $S^{(n)}$ lying outside S' coincident by hypothesis is divided by this line into two parts, one which together with the exposed portion of S' [illegible] continually increasing during the process of variation of $S^{(n)}$ make up a closed surf. separating the interior of both spheres from the whole exterior space not belonging to the spheres, and the second part which, as derived from the portion of the variable surf. that remained inside the closed surf. combined for both spheres during the whole process of variation, must still belong to the interior and hence must be enclosed between the outside of S' and the interior side of the part of $S^{(n)}$ previously considered. Some volume must, therefore, be enclosed between the outside of S' and the outside of the portion of surf. of $S^{(n)}$, last considered.

Describing now a sphere from O with a rad. of the dist. of some pt. in this portion of volume which is greater

than the rad. of the touching sphere $S^{(n)}$, we see that its surface must enclose every pt. on $S^{(n)}$ and hence also the line of contact: it must, therefore, enter into the interior of S' along a line lying on the covered portion of S' and again come out from there by another line lying on the exposed portion of same which is impossible. Hence the contact must be in a single pt.

We say that the rad. of $S^{(n)}$ described from O and touching S' is $= a - b$ and the dist. from O to any pt. in $S^{(n)}$ and in particular to the pt. of contact is $a-b$; the dist. of this last pt. from A being b, the distance from O to $A = (a-b) + b = a$, in perfect agreement with original hypothesis.

If instead of diminishing the radius of the sphere S, we should continually increase it, we would by a reasoning perfectly similar to the preceding come to the conclusion that the space bet. the exterior portion of the surface of $S^{(n)}$ and the interior of S' would continually diminish (and also the exposed portion of the surface of S') and could be made less than any assignable volume, while the two surfaces would have to touch either along a single line or in a single pt. But in the limit it cannot be along a line for then, since $S^{(n)}$ will have to inclose in its interior both the portion of S', which was interior to S at the outset and which kept on increasing during the process of variation of $S^{(n)}$ and also the remaining portion of S' which has been exterior to $S^{(n)}$ and was decreasing during the process) there must exist in the limit a portion of volume contained bet. the sorti-

of surface of S' considered last on the outside of it, and that portion of surface of $S^{(n)}$ which is now separated from ~~does not contain in its interior~~ the first portion of S' by the surface of the other portion of S'. The two portions of surf. enclosing this volume ~~not considered~~ and belonging to the two spheres resp. are separated by the line of contact.

Now if we describe a sphere from O by the dist. from it of any pt. within this space, this sphere will lie wholly within $S^{(n)}$ and must be one of the spheres which have cut S' before reaching the limit. It must, therefore, first issue from that space, but as it cannot cut $S^{(n)}$, it must cut S' in two lines, one on either of the portions of its surf. separated from one another by the line of contact of S' and $S^{(n)}$, which is impossible. Hence the contact of S' and $S^{(n)}$ must be in a single pt.

We then say that the rad. of the sphere described from ~~described from~~ O, enclosing S' and touching it in one pt. is equal to a+b.

Suppose now that we have to add to OA a dist. AB, greater than OA. We describe then from A a sphere with rad. AB; it will enclose O and, hence, at least a portion of S; it may, however, inclose it all. Describe now a sphere S_1 from O with rad. = to AB; it will inclose A and hence, either of S_1 and S' will enclose portions of one another; but neither can enclose the whole of the other, since ~~supposing~~ ~~O on A and A on O~~ as one cannot be greater than the other they would have to coincide, which is impossible, the centers

being distinct. Hence they will increase as previously. Increasing now the rad. of the sphere described from O, we can prove as in the last case that some $S^{(n)}$ will come to touch S' in one pt. B and inclose it all, so that OB will be equal to OA + AB, as previously. We see now that we can add and subtract distances in the same way as abstract quantities.

finition 3) By conceiving the sphere S' described from A to vary continuously beginning from one whose rad. is indefinitely small and passing all imaginable distances, to become indefinitely large, we shall get all possible distances OB, positive and negative whose general expression is $(a \pm x)$; to each such dist. from O corresponds in construction one and only one pt., and the aggregate of all these pts. will evidently form a continuous line, which satisfying fully the conditions necessary for a line suitable to represent the distance magnitude, may be called the distance line or the straight line.

Cor 1 To every distance on the line as measured from O there will correspond a pt. B, at a corresponding distance from A, such that no other pt. in space can have the same distances from O and A resp.

For, if we describe two spheres from O and A with the corresponding distances as radii, these two spheres will have only one pt. in common, namely, the pt. on the str. line; but if there were another pt., i.e., space of the same dist's from O and A, this last pt. would also be on both these spheres, which is contrary to construction.

Cor. II. If A and B and A' and B' are two pairs of pts. at equal distances from one another singly, then the str. lines connecting each pair will coincide with one another (as soon as the congruent pairs (A,B) and (A',B') are made to coincide along the whole of their extent). This follows immediately from the fact that the distance line is unique bet. two pts. of fixed position.

Cor. III. Every pt. not on the distance line bet. A and B or its prolongations is such that its distances from A and B resp. a whole series of other pts. can have.

For since the given pt. is not on the distance line bet. A and B it follows that the two spheres thro. the pt. described from A and B as centers will not touch in this pt, but cut along a line which is the locus of all pts. having the same distances from A and B as the given one: hence the corollary.

Cor. IV. If we imagine a rigid body to be placed so that two of its pts. coincide with A and B, considered previously, ~~to which~~ and (to be) ~~made coincid~~ fixed there, then all the pts. of the body fall in (either of the following) two categories: 1st, a continuous series of pts. coincide with corresponding pts. on the str. line bet. A and B which is fixed in space, as long as A and B are fixed; 2d, the remaining pts. coincide ~~with~~ each with a pt. in space, whose distances from A and B it has in common with every pt. on the intersection of the two ~~imag~~ spheres described from A and

B thro. this pt. It is evident, therefore that any pt. in the body of the second categ. can be displaced along the line of intersection of the two Spheres on which it originally fell, while (A) and (B) are still fixed in A and B, letting all other pts. in the body take care of themselves or rather the rigidity of the body, which consists in preserving the relative distances of the pts. of the body, take care of their individual positions during this displacement. We find then that every other pt. of the second category will have in all its displacements to remain on a corresponding line of intersection of two spheres from A and B; every pt. in the first category, however, will have to remain stationary, since there is no other pt. in space having the same distances from A and B, besides the one with which it coincided originally. These conditions are seen to be perfectly compatible, also, with the relative distances of the individual pts. of the body from one another, as soon as a continuous series of spheres, described from A and B as centers, is imagined together with their mutual intersections for as the pts. of the body ~~are assisted to their~~ move along these lines of intersection, these lines together with their spheres can be conceived to glide upon themselves, never changing their pos. since a sphere is a homog. surf.; hence the mutual distances of the moving pts. are defined alike throughout their motion. Such a motion of the body about a stationary st. line, as we call rotation

The str. line can also be derived, independently of the notion of 2 series of concentric spheres touching one another, in the following manner, which gives us a more concrete picture of the line.

Theorem 3) Take a fixed pair of pts. AB, fig. 1. Then all pts in space can be divided into two categories; to the first belongs every pt. C which has at least one other corresponding pt. C' such that AC' = AC and BC' = BC; to the second category belongs every pt. D whose distances from A and B resp. no other pt. in space can have.

Demonstration. The statement that only these two categories are possible does not need any proof. That both categories actually exist will become clear from what follows.

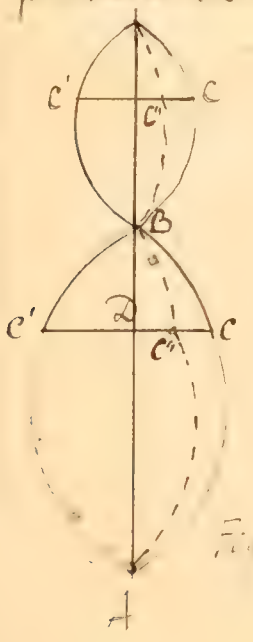

Fig. 1.

Let C be a pt. of the first category and C' its corresponding pt., somewhere else in space. Connect C with A and with B resp. by means of two rigid pairs of pts., (AC) and (BC). Let now the ends A and B of the rigid pairs remain fixed and the common end (C) be free. Then (C) can be made to coincide with C'. It is evident that if ACB represents any line AC'B, which denotes the position of (ACB) when (C) comes to coincide with C', will represent a line perfectly equal to the first; hence, the aggregate of positions of (ACB) as it passes from ACB to AC'B will be a surf., and (C) will pass from C to C'

⊕ for example line $\overline{AB'}$ traverses the volume $ABECBE$ of the volume of this region previously described is shifted so that its back limit (ABE) coincides with ACB' which has originally been its front limit and now constitutes the common contact of the two volumes and the same is also true w. r. to the shifted volume, when this is described by its back limit ($AC'B'$).

in a path $CC''C'$ any pt. of which will be a corresponding pt. w.r. to C or C', i.e., it will be equidistant with there from A and B resp. Now (C') can pass to C, either by retracing its steps backwards, or in some other way; this other way must exist, since if we imagine the surf. $(ACBC''C')$ to be rigidly held only at the pts. A and B, then while the edge (ACB) moves up to $AC''B$, where C'' is on the road from C to C', the edge (ACB) must move away towards some position $AC'''B$, where C''' is on the continuation of the line $CC''C'$, and where the whole line $AC''B$ is either not on $ACBC''C'$ at all, or, if it is there, it must be in the region $ACBC''$, (C') having crossed C and now lying on C''', between C and C''. In the second case, we have proved our assertion; in the first case, we may continue the process until (C') comes finally to C, which must happen some time. For, if we imagine a rigid surface, different from our surf., passing through the moving line (ACB), and of which another limit is (AEB), where (E) is some fixed pt. in this new surf. $(ACBE)$, this last will describe a volume $ACBEC'BE'A$ in the course of motion of (C) from C to C'; and for every addition of a part (CC') to the line $CC''C'$, there will be an addition of a strip of surf. equal to ACC' and an addition of volume equal to $ACBEC'BE'A$. And ⊕ the number n of additions can become greater than

any original quantity it follows that the number of strips of the surf. and the corresponding number of volumes, each equal to the original volume can be made greater than any assigned number; hence, if (C) never comes to C we would get a volume greater than any assigned volume described by the finite surf. (ACBE) which is impossible for we can describe from either A or B a spherical surf. with a radius equal the distance of the farthest pt. in (ACBE) from either of the centers and this spherical surf. will then enclose a larger portion of volume than any volume which can be described by (ACBE), this last remaining always within the spherical surf..

(C) describes a closed hon

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nating the exterior shel

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Cor. I. Revolving a solid, in which a str. line is fixed, about any two pts. in the line which are fixed in space the whole extent of the line will remain fixed in space.

Cor. II. Any portion of a str. line is again a str. line.

For, if C & D lie on a str. line AB, then revolving a solid about AB as a fixed axis, C and D remain fixed; hence the portion of the line bet. C and D remains fixed when the solid revolves about C and D when these are fixed, i.e. this portion is a str. line.

Cor. III. Any portion of a str. line will coincide with any other portion of the same str. line if only their limits coincide, or in other words, a str. line is homogeneous, i.e. any portion can move upon the whole without deformation.

Cor. IV. If we move up the segment AB of a str. line upon itself a distance AA', so that the position of AB at the end of the motion will be $A'B'$, then the whole line $AA'BB'$ is also a str. line.

Fig. 2.

Let us imagine that the line $AA'BB'$ representing the segment AB in its original and final positions is traced upon a solid; then if the solid revolves about $A'B$, BB' remains fixed and also AA'; hence the whole of the line $AA'BB'$ remains fixed when the solid revolves about two pts. in it, i.e. it is a str. line.

It follows that in this way we can

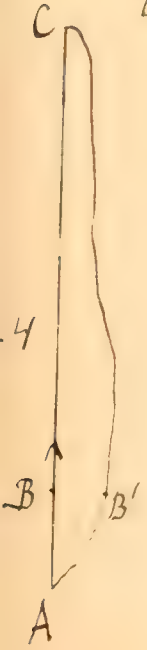
proving a segment of str. line indefinitely only by moving it along itself and its successive prolongations.

Cor V. Two str. lines cannot have a portion in common on one side of a pt. and their segments not coinciding on the other side of the pt., like ABC and ABD.

Fig. 3.

For, otherwise by revolving a solid containing both of them about ABD, fixed, BC would be displaced which is impossible if ABC is a str. line

Theorem 41. A str. line AB, issuing from a fixed pt. A and prolonged indefinitely on the other side of B, will not return to A again after any finite number of prolongations, each equal to AB, however large that no.



For, if this were possible, then taking just half of the whole extent, whose end let be C, we would have two distinct str. lines between A and C namely ABC and CB'A, since CB'A is supposed to be different from ABC, as the pt B is not supposed to retrace its steps backwards beginning from C.

Cor. From this theorem and from the preceding one it follows that two distinct str. lines can have no more than one pt. in common, like AOB and COD; as soon as besides O, another pt. E in COD is made to coincide with E' in AOB the two lines must coincide up to the ends of the smaller segments of either, on both sides of O.

Fig. 5

efinition 4) A pair of str. lines having only one pt. in common are said to diverge and form an angle at the pt C which is the vertex of the angle and the str. lines themselves are the sides of the angle. Such a pair of str. lines is called a crossing pair of str. lines.

It is evident, since it has four segments, having six combinations in pairs, if we leave out the two pairs belonging to the same str. line each, we get only four. the number of angles as defined above We shall later in extension of the definition which will give an indefinite number of angles).

We shall consider at present one angle made only by two segments of distinct str. lines issuing only on one side of the vertex O, like AOB. We imagine this to be a rigid figure in which any two pts. K and L, each on one side, preserve always a constant distance from one another.

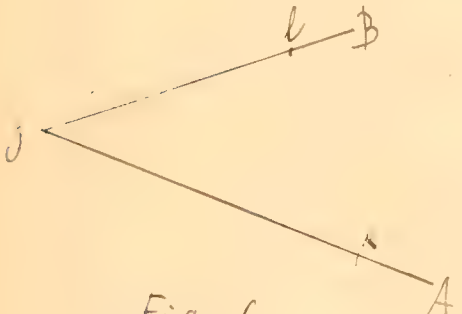


Fig. 6.

The vertex and one side of any other angle $A'O'B'$ can evidently be applied to the vertex O and either side of AOB. If at the same time the remaining side $O'B'$ a pt. in $O'B'$ can be made to coincide with a pt. equidistant from O on the corresponding side of the angle AOB without breaking the rigidity of either of the figures, as explained above, the whole side $O'B'$ will coincide with OB and the angles will be said to be equal. If on the contrary this is not possible the angles are unequal.

Let us see whether there is a way of measuring angles.

orem 5) If two str. lines AOC & BOD intersect so that the adjacent angles AOB and BOC which they form are such that have one side in common) equal to one another, all four angles are equal.

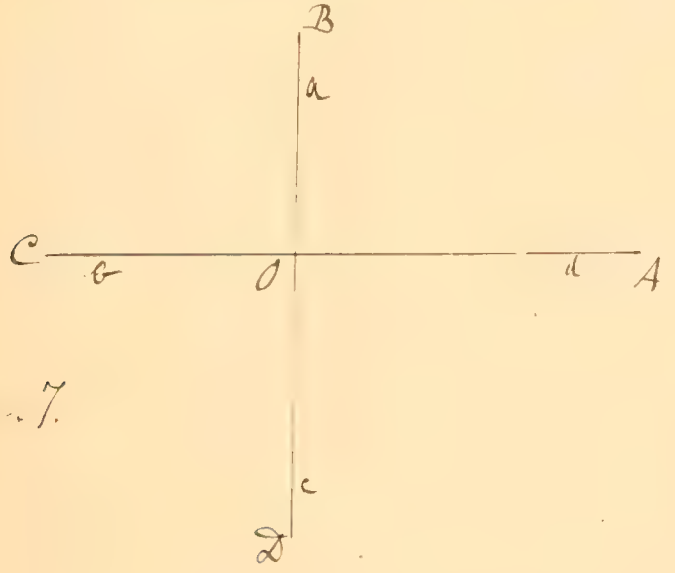


Fig. 7.

For put an exactly equal movable figure, which we denote by small letters, upon the given one, so that ∠aOb of the movable figure shall coincide with ∠BOC of the original figure, then the prolongation of Oc aO, will coincide with OD, the prolongation of BO, and Od, the prolongation of bO, will coincide with OA, the prolongation of CO. Hence ∠BOC = ∠aOb = ∠cOc = ∠COD = ∠cOd = ∠DOA = ∠dOa = ∠AOB, i.e. all four angles are equal.

orem. 5) Each of the four equal angles formed by two intersecting lines is called a right angle and the lines are said to be perpendicular to one another.

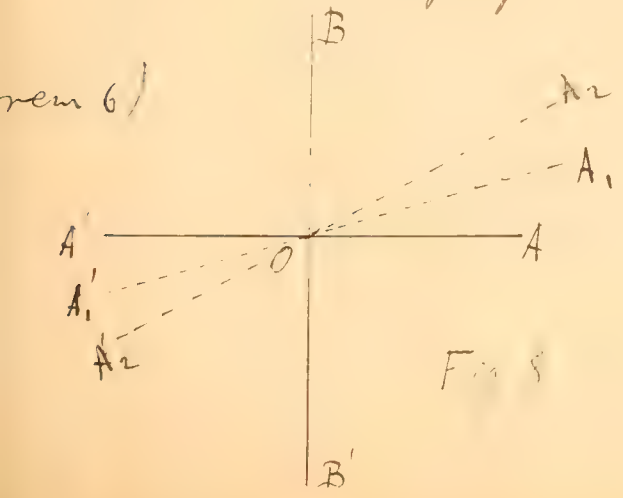


Fig. 8

orem 6) Let the str. lines AOA' and BOB' intersect one another at rt. ∠s. Let one of these be fixed and the other turn around O in space, AOA', say, taking up all possible positions compatible with the rigidity of the figure. Then OA' a

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and B&C fixed, (bc) must somewhere
in space pass ~~through~~ OC, which is supposed to remain
the first occupied; since after (bc) comes into
original position, (bc) will have described
surf.; separating a finite portion of space
ther space. This surf will consist partly
of the plane described by the segment
element Oa_2 in which a is situated,
of the lateral surf described by (ab),
of which will be at some appre-
from O. Hence a sphere described
least dist. from O of any pt. in the
it will enclose an appreciable portion
~~a sphere described~~ while a sphere, described
greater than the greatest dist of the
from O, will enclose in its interior
of OC having some pts. exterior to
surf.; hence a movable pt. on OC, in moving
from a position in the interior portion
position in the exterior portion,
the closed surf.; but as OC cannot
Aa, nor the normal OB in any other
it must pierce the lateral surf. des
(ba) in some pt. c; now, the gener
of this position of bca intercepts OC
∴ &

with the original lower side and vice versa, the ray itself remaining fixed; since in this motion the ray will always coincide with itself, while the normal is displaced and revolves around O into coincidence with its own reversed position.

or II It follows also that a plane divides all space into two equal portions, coincident with one another, as soon as the normal coincides with its reversed position.

Cor. III. The plane described by the segment OA, using the perpendicular OB as an axis is identically the same as that described by the prolongation [illegible] this plane is unique as long as BOB' is fixed.

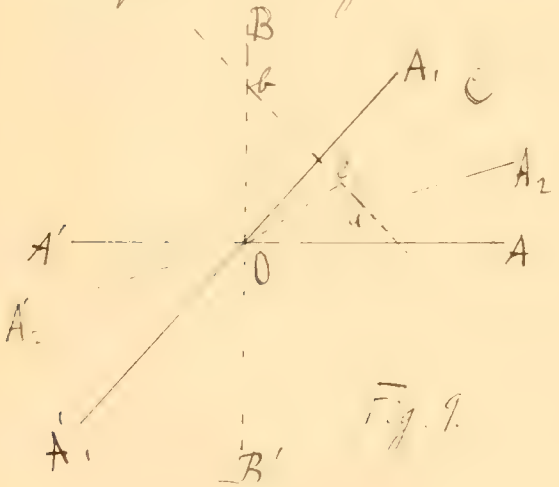

Fig. 9.

: III. Given the plane AOA, whose normal is OB and a str. line OC not lying in this plane, then BOC is not a rt. ∠.

For joining b and a, a pt. on the normal with some pt. on the plane and then revolving ~~BC around OB fixed until OC comes in contact with a pt. c in OC~~ * Let then the element (OA) starting from its new position ~~on which a is situated~~ describe its own pl. and, dragging [illegible] OC in its motion; then this last will describe a surf., homogeneous w. r. to its own motion, which will wholly lie on the side of the pl. AOA₁, where OB is situated. Reversing now the pl., the surf. generated by OC will lie wholly on the side, where OB' was [illegible]

* [illegible]

originally, and will be separated from its original position by double the space bet. its original position and the plane AOA_1. It is therefore not a plane; hence BOC is not a right angle.

Q.E.D.

r. V It follows then that all planes coincide with one another as soon as their normals and origins coincide.

eorem 7) All right angles are equal.

Demonstration. If the vertex of any rt. angle is made to coincide with the origin O of our plane and one of its sides with the normal OB of the plane the other side must coincide with some one of the elements of the pl., say OA, by preceding Cor. hence all rt. angles can be made to coincide, or are equal.

Q.E.D.

The preceding theorems about the properties of a plane and of right angles will suffice to render our notions about angles in general, as geometrical magnitudes, more concrete. First we observe that we have found a natural unit for measuring angles namely the right angle and secondly, we shall soon see how the pl. will afford us a means of comparing all possible angles with our standard unit the right angle. The following preliminary remarks are necessary.

ol. I Any pair of crossing lines will be capable of being applied to the pl. so that the vertex is on the origin and the lines themselves shall lie wholly in the plane.

In fact, one of the two lines can always be brought into coincidence with any ray in the pl. so that the vertex fall on the origin. If now the other line falls upon another ray, the proposition is established. If this other line will not [illegible] The pl. [illegible] applying, we may imagine, if necessary a portion [illegible] and then restored back to its original position; [illegible] common with the pl. fig. [illegible] OB or CC in fig. 1. If now we revolve the plane about the fixed ray with which the first side is in coincidence it will sweep thro. all space during the revolution which brings the normal into its reversed position: hence in time during this revolution it will have to intercept another pt. on the line which has not been in its original and which is now have in coincidence in position; but just as this interception occurs this line will have two pts in common with some ray in the pl. and will coincide with it along the whole of its extent, i.e. both lines of the crossing pair will lie in the pl. Hence all pairs of crossing lines can be made to coincide each with a [illegible] ray in the pl. In other words, all possible angles are to be found among the different angles between the different rays of any plane.

Book II

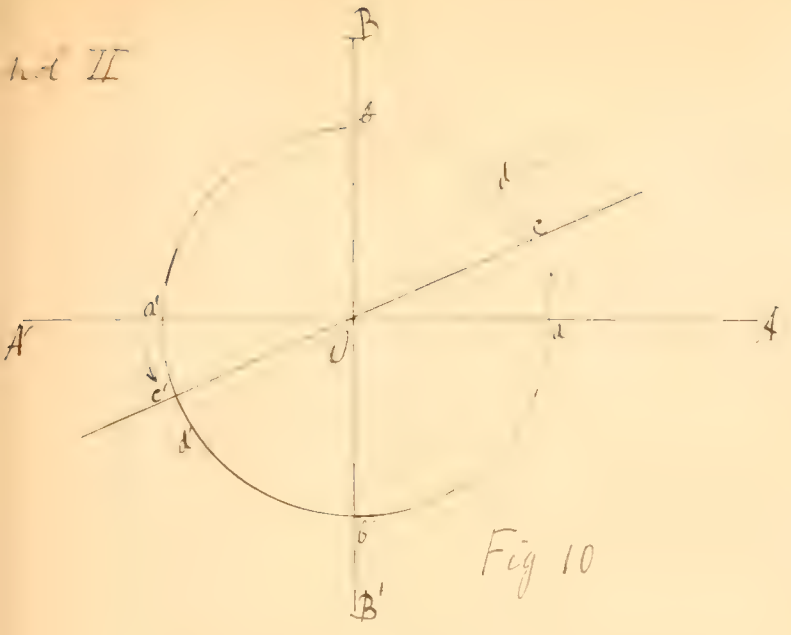

Fig 10

If we apply any two perpendicular lines of indefinite extent to a correspondingly pair of rays in the plane, we see that the whole extent of the plane can from any initial position of a ray, OA say, be divided into four congruent parts, each of which will be enclosed by a right angle. The end of any fixed distance from O, measured along a ray, will generate in its motion around O a homogeneous curve, every pt. of which is equidistant from O. The curve is called the circumference of a circle and O is its center. A portion of the pl. enclosed between any two radii measured one way from outer to circumference, is called a sector of the circle [illegible] enclosed by the circumference and will coincide with any congruent portion of the circle enclosed between two radii making the same angle with one another. It is now evident that, since whenever the angle bet. two fixed radii is equal to the angle bet. two other fixed radii with the sectors of the circle and the segments of the circumferences enclosed by the resp. pairs of radii are congruent to each, the angles can be measured either by the sectors or by the arcs enclosed between their sides,

sector or arc, and a multiple or part of an angle corresponds to the same multiple or part of the corresponding sector or arc. Since every possible position of two intersecting str lines, w. r. to one another, has been proved to find its analogue in some position of two intersecting rays of a plane it is sufficient to investigate these last. Now we can conceive the whole system of displacements possible for a ray in the plane Fig. 19, [illegible] ray $A'OA$, to be bound up with the corresponding displacements of a circle, rigidly fixed to the moving ray and revolving with it about the center, so that the circumference moves in its own trace. Then, we see that if (Oc), one segment of the moving ray, makes in any of its positions an angle AOc with OA, measured by the arc passed over by the moving pt. in the circumference from a to c, then the prolongation of (Oc) must have been displaced just as much on the other side of OA', the prolongation of OA, and must form an equal angle $A'Oc'$, as measured from the fixed segment OA' or along the circumference of the circle from a' to c', in the same sense as ac.

For any fixed pt in the moving circumference must have been displaced an equal arc with [illegible]. So we see that the two prolongations of the same ray lie on opposite sides of the two prolongations of any other ray not [illegible] with it. Since any fixed ray can be taken as the initial ray, and the angle made by the prolongations of the two will be evidently equal to the angle

by the rays themselves. Also, that any pt. in the moving ray will be displaced a quarter, a half, 3/4 and a whole circumference, when the ray is displaced one, two three or four right angles; also that the ray will fall upon its own prolongation for a displacement of two rt. angles and will come to its original position for a displacement of four right angles. It follows then that any ray in any plane makes with the prolongations of another ray two angles one of which is just as much less than a rt. angle as the other is greater, their sum being = 2 rt. angles (acute and obtuse angles). *

Following theorem concerning angles in space becomes at once evident by the application of the angles to the plane.

orem 8) Two adjacent angles, whose two non-common sides form the opposite prolongations from the vertex of the same str. line are together equal to two rt. angles; the two non adjacent angles, made by two str. lines and their prolongations resp. are equal, and all the four angles are equal to four rt. angles.

tion 7) A figure bounded by three str. lines intersecting two by two in three pts. and making three angles, each less than two rt. angles is called a triangle. We must not at the beginning consider a triangle as containing any surf. that may be limited by the sides of the triangle. ⊕ It is evident from the way we constructed a str. line that any three pts. at fixed distances from each other are capable of lying with any

* If two rt. angles are defined as usual.

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rem 10) A str. line having two pts. in common with a plane will lie wholly in that plane.

Demonstration. If the two pts. are upon the same ray the str. line coincides with the ray, i.e. lies in the plane. Suppose, however, that the two pts. lie upon different rays of the pl. Prove that any other pt. of the str. line lies upon some other ray of the pl.

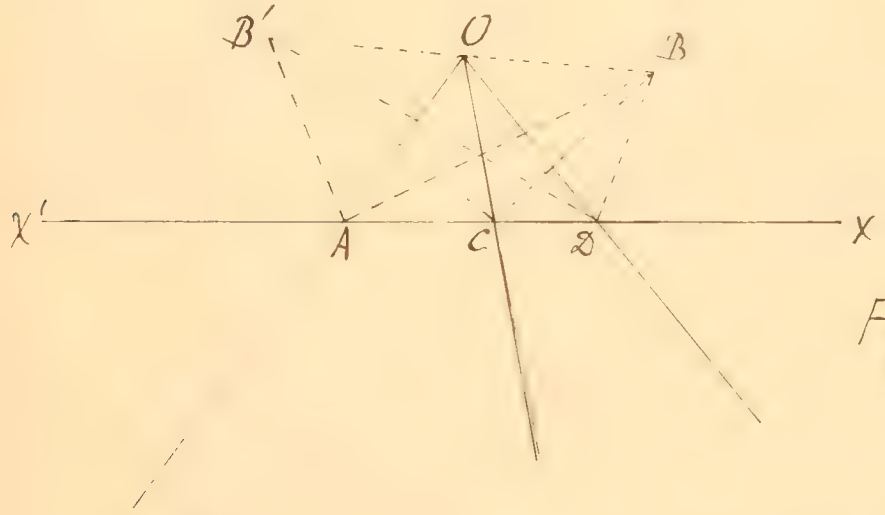

Fig 11

Let BOB', where $= OB - OB'$, be the normal of the pl. ABC, and let the str. line $X'X$ cut the rays OA and OC in the pts. A and C resp.; we have to prove that any pt. D of the line $X'X$ lies upon a ray OD. Connecting OD, we join B and B' resp. with A, C and D

Then $\Delta BOC = \Delta B'OC$ and $\Delta BOA = \Delta B'OA$ by last proposition

$\therefore BC = B'C$ $BA = B'A$, whence $\Delta ABC = \Delta AB'C$ (by conv. to definition 7).

Whence $\angle BAD = \angle B'AD$ and $BA = B'A$;

$\therefore \Delta BAD = \Delta B'AD$ and $BD = B'D$;

$\therefore \Delta BOD = \Delta B'OD$, i.e. $\angle BOD = \angle B'OD$

$\therefore$ [illegible] $= rt. \angle$ [illegible] Theorem 1.

OD is therefore a ray. Now, since this is true for any pt. B in the line AA', the whole of it must lie in the plane. Q.E.D.

Defin. 8) A plane can now be redefined as the surface in which every str. line lies wholly, if it passes through two of its pts.

It immediately follows that any angle can be made to lie wholly anywhere in the pl.; and so also a triangle, — for one side of the triangle will lie ~~in the~~ wholly in the pl. as soon as its two ends lie in the plane; now if also the third vertex is brought into coincidence with some pt. in the plane, by revolving the triangle around the side held fixed in the pl., the other two sides will likewise come wholly into the pl.

Defin 9) The normal is said to be perpendicular to the plane because it is perpendicular to every str. line passing thro. it and lying wholly in the plane.

Theorem 11) A str. line perp. to any two intersecting str. lines at their pt. of intersection is perpendicular to the pl. in which these str. lines are situated. The str. lines may not be the rays of the original construction of the pl., but any two str. lines intersecting there, which were proved to lie in the plane.

the proof, and proved that given in Book XI, proposition IV of his Elements.

Cor. I. An immediate consequence is that at every pt. of a plane we can erect a perp. to the plane.

For if K is such a pt., not the origin of the original construction, and LKm and KN are any two perpendicular str. lines thro. K in the same plane then fixing the line LKm and revolving (NK) about it, the amount of one quadrant, into the position of (N'K), this last is now perp. to Km and KN; hence it is perp. to the plane in which they are situated. In other words, KN' may now be regarded as a normal and all str. lines in the given plane thro. K as the rays.

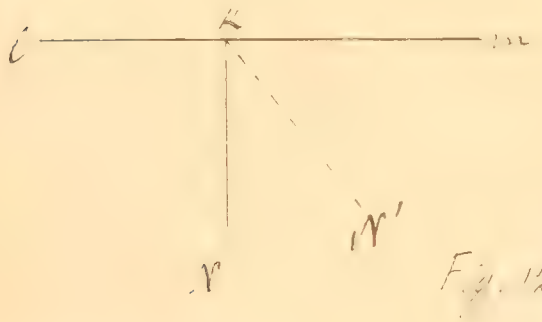

Fig. 12

Cor. II. Since any two planes coincide if some of their origins and normals coincide it follows that a plane will coincide with itself when the origin is displaced in any manner whatever to any other pt in the same plane, provided the normal at the origin is made to coincide with the normal at the pt. to which the origin is displaced; also that the origin or indeed any pt in a pl. may be made to describe any curve continuously describing any curve in the plane the whole plane remaining unaltered in shape or position, also that the plane may be turned

upside down, shifted upon itself in any manner as a whole
whatever without altering its position or shape
(Leibnitz).

At this stage of our investigation, our elementary figures, viz. the angle, the triangle and the circle can be made more concrete. Since each of these figures (can be made to lie) wholly in a plane, we may suppose their boundaries to limit corresponding portions of a pl.. This is the reason of their being called plane figures in contradistinction from those which cannot be made to lie wholly upon a plane. Thus, a network of str. lines and circles may be conceived to cover the plane figure, and in the plane itself, crossing and recrossing one another in all possible ways, since every pt. in a plane may be conceived as an origin of rays and angles and as a center for the circumferences of the circles, described in fixed pts in the plane (w.r. to the plane) etc. Geometry makes use of this conception unvariably, in the way of a fiat, "Describe a circle from such and such a pt. as center and with such and such a distance as radius &c., just as it does w. r. to homogeneous space in general.

The following propositions of the first book of Euclids Elements can now be proved very rigorously either by Euclids Demonstrations, or in a more elegant way like the one used by Legendre and his followers. since none of the hypothesis whether tacit or explicit which Euclid assumes in the shape of postulates definitions

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(Theorem 12) The sum of any two interior angles of a triangle is less than two right angles.

$a + b < 2$ rt Ls.

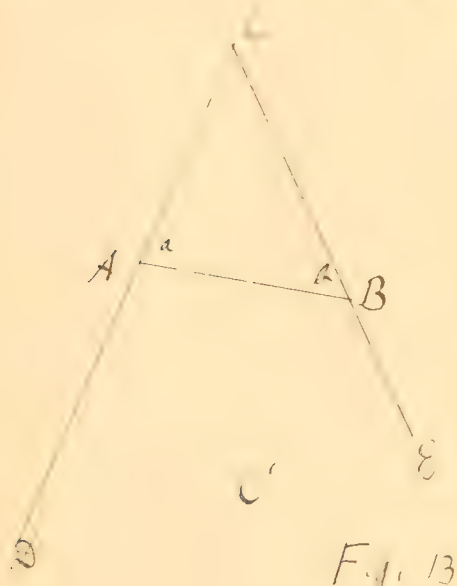


Fig. 13.

Dem. If $a+b \geqq 2$ rt Ls, then because $\angle ABE + b = 2$ rt Ls (Theorem) it follows $a+b \geqq \angle ABE + b$, and $a \geqq \angle ABE$. For a similar reason $b \geqq \angle DAB$. Now, applying a [illegible] part of the diagram $\angle ABE$, to the upper ABC so that $\underline{A}$ shall fall upon $\underline{B}$ and $\underline{B}$ upon $\underline{A}$, and the angle $\underline{a}$, being $\geqq \angle EBA$, shall coincide or inclose the latter and $\underline{b}$ coincide or inclose $\angle DAB$, AD will lie either on the side BC or within the angle $\underline{b}$, and BE either on AC or within the angle $\underline{a}$. In ~~every~~ either case AD and BE will intersect ~~again when~~ at $\underline{C}$, the vertex of the triangle, or in a pt. within the triangle. And when returned to their original positions they, while making the prolongations of CA and CB resp., will intersect at some other pt. C', below AB, or in other words, the two lines CAD and CBE will intersect in two pts $\underline{C}$ and C', which is impossible (Theorems 2 and 3 and corollaries). Therefore, it is impossible that [illegible] $a+b \geqq 2$ rt Ls.

Q. E. D.

Cor I. If in the formula $a+b \leqq 2$ rt. $\angle$s we separate the case of equality, namely $a+b = 2$ rt. $\angle$s, we shall have $a = \angle ABE$ and $b = \angle DAB$; and the two str. lines making such angles with the secant cannot meet either below or above the secant, for in either case they would have to meet simultaneously also on the other side of the secant which is impossible. Therefore if two str. lines intersected by a third make with it two interior angles on the same side of the secant equal to two right angles, they cannot meet, even if produced indefinitely both ways.

Cor. II. Any exterior angle of a triangle is greater than either of the interior and opposite angles.

Because any of the interior angles with its adjacent exterior angle equals 2 rt. $\angle$s (Theorem 8), while with either of the opposite interior angles it is less than 2 rt. $\angle$s; hence either one of the last mentioned is less than the exterior.

Cor. III. If one of the interior angles of a triangle is a rt. or obtuse angle, either one of the remaining two is less than a rt. angle; therefore, in a triangle there can be no more than one right or obtuse angle and not less than two acute angles.

Cor IV. A perpendicular is the shortest line from a pt. to a str. line. (Prop. XVIII Euclid's Elements). A perp. is therefore assumed to represent the distance from a pt. to a str. line.

10) A quadrilateral is a figure bounded by four sides, containing four angles. A skew quadrilateral is one bounded by the segments of four str. lines of fixed length each, between four pts: the vertices of the angles, each vertex being connected only with two adjacent ones. A plane quadrilateral is one that can be placed wholly in a plane. We shall have occasion to use quadrilaterals of fixed sides only & not of fixed angles. Of course these are, so to speak, non-material ones, i.e. bounding no fixed plane area, and in such the relative distances of the four vertices are fixed only for the four out of all 6 possible pairs, or stetuting the ends of the four sides resp. Thus three distances bet. three pts., AB, BC, CA must necessarily fix all distance relations bet. three pts. a distance being a relation bet. one pair (no. of combinations of 3 different things taken 2 at a time), only 2 distances will be sufficient to fix uniquely the distance relations bet. four pts. (no. of combinations of 4 different things taken 2 at a time); hence four distances are insufficient in the case of four pts. If, however the quadrilateral is restricted to lie in a plane we will have an additional relation and only one additional distance, as one diagonal the distance bet. either pair of the opposite vertices will be sufficient to determine the complete form of the quadrilateral. The impossibility of a quadrilateral is expressed by saying that a quadrilateral is rigid.

+ Cor II. It also follows that in a pol. quadrilateral with equal opposite sides, the two non-opposite sides and the angle included by them are sufficient to determine the whole quadrilateral.

theorem B). In any quadrilateral, which has its opposite sides equal, the opposite angles are equal.

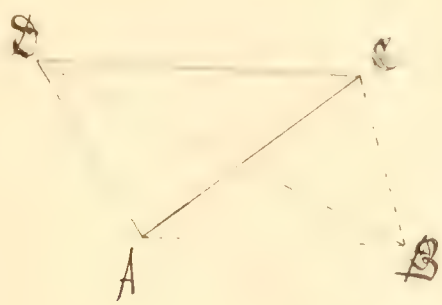


Fig 14

Let AB = DC, AD = BC
then joining AC and

$\Delta ABC = \Delta ADC$

and $\Delta ABD = \Delta DCB$;

$\therefore \angle ABC = \angle ADC$

$\angle BAD = \angle DCB$ Q.E.D.

Cor I. It also follows that

$\angle ABC = \angle BDC$ and $\angle ADB = \angle CBD$;

Similarly $\angle CAB = \angle ACD$ and $\angle ACB = \angle CAD$,

i.e. the angles made by the same diagonal with the two opposite sides are equal.

theorem C). Two plane quadrilaterals, having three sides and two included angles in the one respectively equal to the corresponding three sides and included angles in the other are equal to one another.

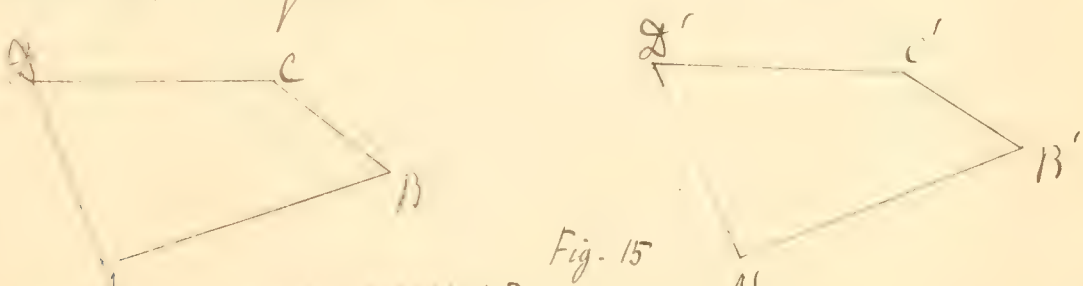


Fig. 15

If DA = D'A'
AB = A'B',
BC = B'C',
also $\angle DAB = \angle D'A'B'$
$\angle ABC = \angle A'B'C'$,

then by superposition we
that the two quadrilaterals co

(Theorem 15) In a plane quadrilateral two of whose opposite sides are equal and in which the interior angles made by these sides with any one of the remaining sides are supplementary, this last cannot be greater than the side opposite

Fig. 16

Let $OA = O_1A_1$,
$\angle AOO_1 + \angle OO_1A_1 = 2$ rt $\angle$s;
then $OO_1 \nparallel AA_1$.

For, let $OO_1 > AA_1$;
then, since $OA = O_1A_1$
and $\angle AOO_1 = \angle A_1O_1O_2$ [illegible]
$= \angle OO_1A_1$, we can apply the lower side ~~of the quadrilat~~ of the quadrilat. OA to its upper side O_1A_1, and we get $A_1A_2 < O_1O_2 = OO_1$.

Now, suppose $OO_1 - AA_1 = l$, some length; then every time O is transposed a dist. OO_1 along the str. line XX', A is transposed a dist. $OO_1 - l$. Let $OA = ml + l_1$, where m is a positive integer and l_1 is either zero or a length less than l. Then applying this quadrilateral to itself along the same str. line XX' $2(m+1)$ times, we get

$OO_{2(m+1)} = 2(m+1)OO_1 = 2(m+1)AA_1 + 2(m+1)l$
$> 2(m+1)AA_1 + 2ml + 2l_1 - 4A_1A_2 + A_{2(m+1)}A_{2(m+1)}$,
i.e. the str. line bet O and $O_{2(m+1)}$ is greater than

the broken line bet. these same two pts, which is absurd (Euclid XI, Book I). Hence $OO_1 \neq AA_1$. Q.E.D.

Cor The sum of the three angles of a triangle cannot be greater than 2 rt. $\angle$s.

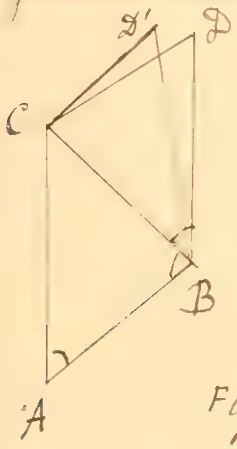


Fig. 17

For, if ABC is such a triangle, then applying to it an equal triangle BCD, where $BD = AC$ and $CD = AB$ we get $\angle DBC + \angle ABA + \angle BAC$
$= \angle C + \angle B + \angle A > 2$ rt. $\angle$s; therefore, if $\angle D'BC + \angle B + \angle A = 2$ rt $\angle$s and $D'B = DB$, we get
$\angle D'BC < \angle DBC$ i.e.
$CD' < CD$ (Euclid I, XI): we have also $D'B = CA$ and $\angle D'BA + \angle BAC = 2$ rt. $\angle$s i.e. $D'BAC$ is just such a plane quadrilateral as was discussed in the theorem and $CD' < AB$, which is impossible; hence the sum of the three angles of a triangle cannot be greater than 2 rt. $\angle$s. Q.E.D.

Theorem 6 An immaterial quadrilateral whose opposite sides are equal can change its angles thro. positive magnitudes, in other words it is free to 'rack', so that the middle pts. of one pair of opposite sides shall always be at a distance from one another equal to the other pair of opposite sides.

Let $AB'CD$ be a quadrilateral whose opposite sides are equal, and E and F the middle pts of AD & BC resp

Observe that if AB and CD, that only the distances AB and BD, on the one hand, and AD and BC, on the other, are equal, no further restrictions being imposed on them, they need not always lie in one plane, if this is required by the conditions of the relative motion to which they are subjected, so that any pair may be in another plane from the remaining two, so that B, for instance, is free to move in one path or other situated in a sphere around C, this last being free to move in a sphere about D and this one about A, provided the four distances do not change during these motions. Initially the distance EF is not given, but this is the very thing we are determining it. At least in one of the infinite number of relative positions capable of being occupied by the ~~distances~~ pts. A, B, C, D, as just defined the distance EF will be equal to either of the sides, AB and DC. This will, evidently, be the case when all four pts. B, C, A, D will fall upon the same str. line, say upon B', C', A', D' resp., where $B'C' = 2BF = 2CF = A'D' = 2AE = 2ED$, and $AB' = AB$, $DC' = DC$.

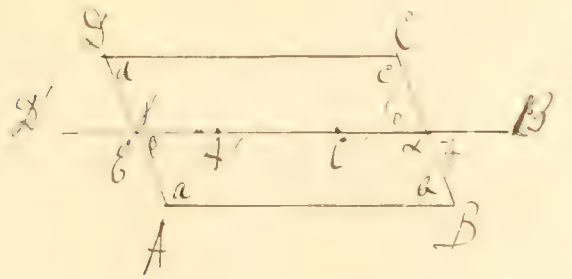


Fig. 18

For $\angle B = \angle D$, always, (Theorem 13) hence when the first becomes zero the second must also become zero that is, when BC falls upon ~~coincides with~~ AB, AD falls upon DC; hence $A'B'$ will be in the same str. line with $B'C'$, when and only when $A'D'$ is in the same str. line with $D'C'$, in other

words B' will be in the same str. line with $A'C'$ and D' will be in the same str. line with these. Hence, F and E are the middle pts. of $B'C'$ and $A'D'$ resp., we have then $EF = EA' + A'B' - FB' = \frac{1}{2} A'D' + A'B' - \frac{1}{2} C'B' = A'B' + \frac{1}{2} C'D'$, since $\frac{A'D'}{2} = \frac{C'B'}{2}$. In this position, then, let EF be hinged to the ends of a str. line of fixed length, or let E and F become fixed so that BC singly may revolve around F in a sphere and AD singly, around E in a sphere. I say that the quadrilateral may be brought into coincidence with a single str. line and may assume all values for any of its angles provided the equality of opposite angles is preserved not only in the whole quadrilateral, but also in the partial quadrilaterals. For, if not, the reason of this must be sought in the mutual geometrical relations of the distances which are the only parts of our immaterial quadrilateral given, including now the additional distance EF i.e. the middle pts of AD and BC, that is to say these distances must be incompatible with an angle β other than zero, so that an attempt to change this value would involve a contradiction in theory and hence a break in the distance relations in practice. Calling the interior angles of the two partial quadrilaterals as in the figure, we get 6 independent relations bet. the angles which, on the supposition that the motion sought is possible, must hold for all values of the angle β, the value $\beta = 0$ included; we have also in addition 2 independent relations derived from the supposition that BF and AEB are to remain str. lines in the whole

course of motion. We have thus enough and just enough relations to determine each angle, if such a determination is at all involved in the mutual relations of the angles and distances. For if such motion were impossible, these relations would have to lead us to a contradiction for any other value of ξ than $\xi = \alpha$. Now, instead of leading to such a contradiction, the first 6 relations give us, as an immediate consequence a new relation verified by the remaining two, showing that our supposition does not involve any contradiction and that such a motion can be realized. The relations referred to are:

$$\left.\begin{array}{l} 1)\ \alpha = a \\ 2)\ \gamma = c \\ 3)\ a = c \end{array}\right\} \therefore \alpha = \gamma \quad \text{I}; \qquad \left.\begin{array}{l} 4)\ \beta = b \\ 5)\ \delta = d \\ 6)\ b = d \end{array}\right\} \therefore \beta = \delta \quad \text{II}.$$

Besides,

$$7)\ \alpha + \delta = 2\text{rt. Ls}$$
$$8)\ \beta + \gamma = 2\text{rt. Ls}$$

Combining I and II we get $\alpha + \delta = \beta + \gamma$ III, which agrees with 7) and 8). Of course I + II can be realized without the relations 7) + 8). But the argument is thereby not invalidated that the motion presupposed in the theorem cannot be impossible under the only conditions given. It may, of course, be impossible under certain additional conditions, like the restriction of the lines to slide on certain surfaces, but such a restriction is excluded in the given conditions where only abstract distances are concerned. The motion itself may involve certain positions as possible and others as

impossible, which we shall presently proceed to investigate only in so far as is necessary for our purpose.

Remark. Such a motion would also be possible even in the case of spherical arcs of great circles, provided these arcs are at liberty to move out of the surface to which they belong initially; but is preceded an additional condition is not put upon the motion that every pt. of all 5 must remain at a constant dist. from the center of the sphere to which they belonged when the spherical angle δ was zero. In other words, in order that such a motion be realized for arcs of great circles of the same sphere, the planes determined by the arcs in motion can have a common pt. of intersection only when the spherical angle of two of the linked arcs is zero or π, i.e. when they coincide with one great circle; for all other values of the angles these planes cannot intersect in the same pt., i.e. the arcs must go out of the common surface to which they originally belonged, as can easily be proved [illegible] the science of [illegible] to a higher stage of geometry to which the treatment of spherical geometry properly belongs. A similar remark holds w.r. to the geodesics upon a pseudosphere. It seems that proceeding from these considerations it ought to be possible to prove that whenever the normals to an homogeneous surf. meet in a pt., or what is the same thing whenever the surf. has constant positive curvature, there must be an excess of the sum of the three angles of a triangle over 2rt. angles, [illegible]

the normals lie in different planes, or the homog. surf. has constant neg. curvature, there is a deficiency of the sum of the three angles of a triangle from 2π.

Analytically, the proof of the possibility of such a motion as defined in the theorem is obtained also by showing that if we add to the above 8 relations an arbitrary relation $\delta = 0$, the determinant of the 9 linear eqs. in 9 variables, when the equations are made homogeneous, vanishes identically, proving that $\underline{\delta}$ can have any value. The determinant in question is, if $a, \alpha, b, \beta, c, \gamma, d, \delta, 1$ is the order in which the variables are written, as follows:

$$\begin{vmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -\pi \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -\pi \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\theta
\end{vmatrix} = 0$$

Q.E.D.

Theorem 17) Lemma. Given an angle and a str. line thro. its vertex not in the plane of the angle, the sum of the angles made by this line with the sides of the angle is greater than the angle.

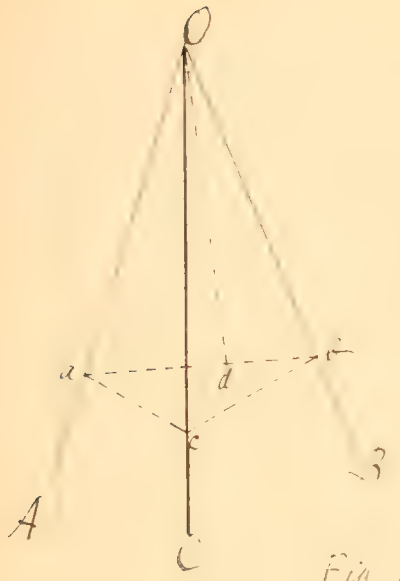


Fig. 17

The theorem needs proof only for the supposition that $\angle COA$ and $\angle COB$, taken singly, are each less than $\angle AOB$. Connect any two pts $\underline{a}$ and $\underline{b}$ on the sides of the given angle AOB by a str. line ab; this last will be in the plane of the angle (Theorem 10), and hence OC, the line thro. $\underline{O}$ outside the pl., cannot meet ab. Pass now a line Od in the pl. AOB at an angle aOd, equal to $\angle AOC$ and cutting ab in some pt. $\underline{d}$, which it must do, since it cannot cut either Ob or Oa again (Theorem 4 and cor.), and because the str. line from O will pierce a sphere inclosing the whole triangle aOb (Theorems 2 & 3 and corollaries). Lay off on OC a segment Oc, equal to Od, and join ca, cb; then $ca = da$ (Theor. 1), and in the triangles aOc, aOd, cOb, dOb (Euc. I, IX), hence $\angle cOb > \angle dOa$ (Euc. XXIV). $\therefore \angle aOc + \angle cOb > \angle aOd + \angle dOb$, or $\angle AOC + \angle COB > \angle AOB$.

Q.E.D.

Theorem 18) In the motion considered in theorem 16) the four sides of each quadrilateral will at any instant be in one plane

For, from I $\alpha = \gamma$ and II $\beta = \delta$, combined with i) $\alpha + \delta = 2$ rt. $\angle$s, it follows that $\gamma + \delta = 2$ rt. $\angle$s. Now, joining FB, we find that if CF is always in the plane of $\triangle AFE$ then CD, FB & hence AB are in the same plane, and the theorem is conceded; but if CF is

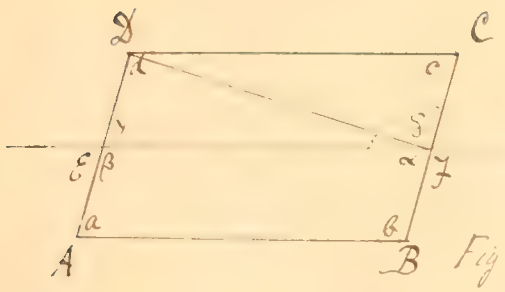


Fig 20

sometimes out of the pl. DFE, then LCFE determines a plane different from pl. DFE, that is DF is not in the plane of LCFE; hence a preceding theorem LCFD + LDFE > LCFE, or LCFD + LFDC > δ (Theorem 13), and since c = γ, we get c + LCFD + LFDC > γ + δ, or 2rt∠ + LCFD + LFDC > 2rt∠, which is impossible (Cor. to Theorem 15), Hence CF is always in the pl. DFE and the theorem is proved as before.

Q.E.D.

Cor. I. It follows that any isosceles quadrilateral with equal opposite sides can move in a plane so that one of its angles assume any value we please and that the middle pts. of one pair of opposite sides constantly remain equal to either of the remaining pair of opposite sides.

Distance bet.

II. It is evident that the sum of the interior angles and hence also of the exterior angles adjacent to the same side in any of the positions of such a moving quadrilateral is equal to 2rt.∠s since 2rt∠s = (α + β + γ) = d + c = a + b = b + c = c.

Theorem 19) The sum of the interior angles adjacent to the same side of any pl. quadrilateral with opposite equal sides is equal to two rt. angles

For an immaterial quadrilateral, having sides of equal lengths with the given ones, can assume such a position in a plane, that the angle inclosed between any two of its non-opposite sides be equal to the corresponding angle in the given quadrilateral, the sum of its interior angles, adjacent to the same side, being equal to two rt angles; hence (by Cor. II. to Theorem 13) it follows, that the sum of the interior angles, adjacent to the same side in our given quadrilateral is likewise equal to two rt. angles.

Q.E.D.

Sum of interior angles.

For an immaterial quadrilateral, having sides of equal length with the given one, can assume such a position that the sum of its interior angles adjacent to the same side be equal to two rt. angles, and that the angle included between any two of its non-opposite sides be equal to the corresponding angle in the given quadr.; hence (theorem 14), since it can be made to coincide with our given quadrilateral, that is the sum of the interior angles adjacent to the same side in our given quadrilateral is likewise equal to two right angles.

Q.E.D.

Cor. I The sum of the three interior angles of any triangle is equal to two right angles.

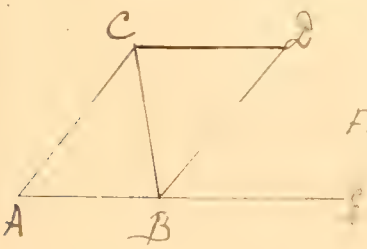


Fig. 21

For, if in the triangle ABC we produce AB to E, $\angle CBE > C$; hence, making $\angle CBD = \angle BCA$, BD falls within the angle CBE (Theorem III Cor. II); making $BD = AC$ and joining CD, we have $\Delta CBD = \Delta BCA$ and $CD = AB$ (theorem 1), hence, we have a pl. quadrilateral with equal opposite sides; therefore, $\angle CAB + \angle ABD = 2$ rt. $\angle$s,

$$\therefore \angle A + \angle B + \angle C = 2 \text{ rt. } \angle\text{s}.$$

Cor. II The exterior angle of a triangle is equal to the two interior and opposite angles.

Cor. III The $\angle$s of a triangle is equal 4 rt. $\angle$s since it can be divided into two triangles having their $\angle$s coincident with the four given $\angle$s.

Theorem 20) Two str. lines perpendicular to a third one in the same plane with it:

1) Will both be perp. to any other perpendicular drawn from any pt. in the one to the other;

2) Will both have the same inclination towards any secant to both;

3) Will be everywhere equidistant from one another, i.e. any perp from one to the other will be of the same length with any other.

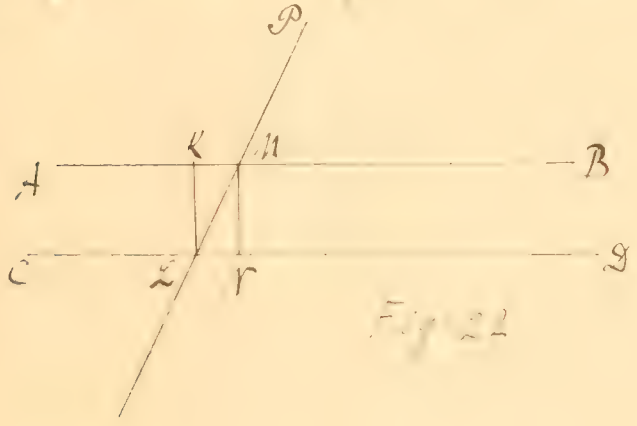


Fig 22

Demonstration.

1) Let AB, CD be both perp. to the same str. line KZ in the same plane and draw any other perp. MN, from any other pt. M in AB to CD and join ZM. The sums of the interior angles of each of the two triangles are equal to two rt angles singly, and together they equal to four rt. angles. But the six angles of both triangles make up together the four angles of the quadrilateral KZNM. Of these last the angles K, Z and N are each of them a rt. angle (by construction); therefore ∠M is also a rt. ∠, and hence any perp. to CD, from any pt. in AB, will also be a perp. to AB. For the same

reason any perp. to AB, from any pt. in CD, will also be perp. to CD

2) In the same diagram, if PLQ is a secant to AB and CD which are both perp. to a third line in the same plane, then draw from L a perp. LK to AB and from M, a perp. MN to CD. The first will also be perp. to CD and the second to AB (section 1 of our proposition) Now in the triangle LMN we have $\angle LMN + \angle MNL = 2rt.\angle s - \angle N = rt.\angle$ (cor. I to theorem 19); besides $\angle LMN + \angle LMK = M = rt.\angle$;
$\therefore \angle LMN + \angle MNL = \angle LMN + \angle LMK$; $\angle MNL = \angle LMK$, or $\angle PLD = \angle PMB$ (Theorem 8); i.e. both AB and CD have the same inclination towards any common secant.

3) If KL, MN are any two perpendiculars to both AB and CD which are situated in the same pl., and ML joins the opposite angles K and L, then $\angle MNL = \angle MKL$; $\angle LMN = \angle MLK$, (section 2 of our propos.) and ML is common to both triangles; therefore $\Delta LMN = \Delta MLK$, and $MN = KL$, - and as the same reasoning applies equally to any other two perpendiculars to both, all the pts. of either AB or CD are at equal distances from the other

Q. E. D.

Definition 11) Two str. lines perp. to a third in the same pl., as having the same inclination towards any common secant and being everywhere equidistant

Cor. III ... (on the same side of) ... the given str. line we get a pol quadrilateral which is congruent with its duplicate turned over so that the equal perpendiculars become interchanged in position; hence the angles opposite the given right angles are equal, and as their sum equals 2 rt. ∠s (Theorem 17, Cor III), each is equal to one rt ∠, that is the line, connecting the pts. and the given line are both ⊥ to the same str. line in the same pol. with these, hence they are parallel.

Cor. IV ... two str. lines in the same pol. having same inclinations towards a common secant are parallel. ... the given lines is constructed two. the intersection of the other with the ... (Section

from one another are said to be parallel to one another, meaning beside one another, or going in the same direction and being everywhere at the same distance from one another.

Cor. I Any perp. to one of a pair of parallel lines in the same pl. with both, if produced indefinitely, must meet the other, at right angles

For, if from the pt. of intersection with the first a perp. be drawn to the other it must also be perp. to the first (Section 1) and therefore coincide with the perp. to the same previously drawn.

Cor. II. From any given pt. without a given str. line only one parallel can be drawn, namely, that line which is at rt. angles with the perp. from the pt. to the given line. * See back of [illegible] page

Theorem 21 Not more than Two pts. of equal distances from a given str. line can be situated in another str. line which is in a different pl. from that passing thro. the first and one of these pts.

Demonstration Let AB, CE be two str. lines of which EC is not in the pl. BAC passing thro. the line AB and the pt. C. The distances of the pts C and E from AB i.e. the perpendicular lines EB and CA drawn from these pts. to AB are equal.

no other pt. on EC can be of the same distance from AB.

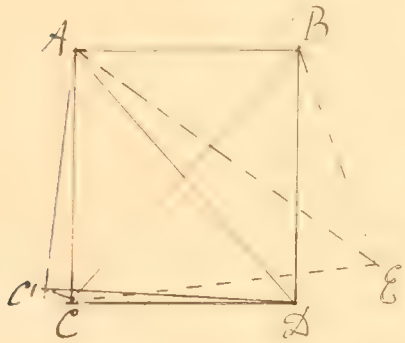


Fig 23

For produce the plane BAC indefinitely, & BC and draw in it CD ⊥ AC ∴ ∥ AB (Defin. 11) and BD ⊥ AB; it will meet CD (Theorem 20, Cor. I) and BD = AC = BE (Defin 11). Join AD, BC, EA. The right angled triangles ABE, ABD and BAC are equal (Theorem 9); therefore EA = DA = CB and ∠EAB = ∠DAB = ∠CBA. Also, since EA is not in the pl. CAB ∠CAE + ∠EAB > ∠CAB (Theorem 17), or ∠CAE + ∠EAB > ∠CAD + ∠DAB; hence ∠CAE > ∠CAD. Now, the triangle EAC must have ∠ECA < rt.∠. For, if we imagine it removed from its original position and applied to △DAC in such a way that EA shall coincide with its equal DA — the other side inclosing the greater angle EAC will fall without the smaller angle DAC and will take the position AC' as in the diagram; and joining CC' we obtain an isosceles triangle ACC' of which the angles C and C' at the base are equal and each less than a rt ∠ (Cor. to Theorem 9 and Cor. III to Theorem 12). Then DC and CC' form an angle DCC' < 2 rt. ∠s having its vertex towards A and therefore, C'D is situated more towards A than C'C, and ∠AC'D < ∠AC'C < rt. ∠.

Now any pt. on the prolongation of EC to the left must also be without the pl. DCA since one part EC can not be without and the other within the plane DCA (Ax.), and any str. line connecting C and another pt. of the same distance from AB without the pl. DCA makes with CA an angle < rt∠ and, consequently cannot be the prolongation of EC since such prolongation forms with CA an angle > rt.∠ (Ax.). Therefore, no pt of the same distance from AB with C can be on the prolongation of EC to the left. For a similar reason, no pt. of the same distance from AB with the two given pts. can be on the prolongation of CE to the right, because CE is not in the plane AEB passing thro AB and the pt. E. Neither can all of the equal distances from AB be situated on CE, bet. C and E, since, then, either E or C would be on the prolongation of a str. line connecting two pts of equal distance from another str. line, one of the pts. connected being in a different plane from that passing through the other pt. and the other str. line, which has been demonstrated to be impossible. Therefore, no other pt besides E and C on the same str. line with them, or on its prolongations, is possible, of the same distance from AB as E and C

Q. E. D.

Cor. I Hence, any str. line in space which connects three pts. at equal distances from another str. line must lie wholly in the plane passing thro. the other and one of its own pts. and, therefore, is parallel to the other and cannot [illegible]

in Theorem 20 with ~~if~~ used as a parallel str. line is true also of any str. line in space having these pts. at equal distances from a given str. line.

Cor. II It also follows that if two equal lines intersect two others in four pts. and three of these lines are at right angles to one another, all four are in the same pl. and, therefore, are parallel, each pair singly.

Cor III. A parallel can also be defined as the locus of all pts. equidistant from ~~a~~ another str. line, and ~~of which~~ ~~that the are in one str. line~~ two given pts.

The definition of alternate, interior & exterior angles of a line intersecting two others in the same plane is as usually given. If the definition of parallels is taken provisionally in the sense in which we have defined them, we see that we have proved Propositions XXVII, XXVIII, XXIX of Euclid, also XXXII and some others of which a no. of corollaries can be drawn w.r. to the conditions of parallelism of str. lines in space and which we will not give here. We are now in a position to prove propositions XXX XXXIII and XXXIV of Eucl. using his proofs word for word. This will now enable us to reduce Euclid's famous Axiom XI and thereby extend the definition of parallelism to any two lines situated in the same pl. and ~~never~~ not meeting one another at a finite distance.

Theorem 22) Two str. lines in the same pl. of which one is perp. to a third one and the other makes an acute angle with it, if sufficiently produced on that side of the secant where the acute angle is situated must meet one another somewhere.

Fig. 24.

Let DC be perp. to AB and AE make an acute angle with AB at A, then AE and BC if produced towards the opening of the acute angle, will meet in some pt.

Dem. Take some pt a on AE, draw a perp. from it to AB (Euel. propos. XII) This perp. will cut AB in some pt L, between A and B, and meet on its production, R. Then in the $\triangle$ LBR, LBA being according to supposition an acute angle, will be greater than a right angle (Theorem 12, Cor II) which is absurd. Let now Al be contained in AB m times (m being a whole number), & m times with some remainder less than Al. Take upon AE beginning from A and proceeding towards E m+1 parts equal to Aa namely Aa, ab, bc... and let f be the end of the last part. Draw now perpendiculars aa1, bb1, cc1... ff1 to

also be parallel to "AB" and to each other (Defin 11).
Draw also bm, cn, ... fq parallel to BC, AD, dl — which will all meet at right angles (if the perpendiculars to AD (Cor. I to Theorem 20). Let the vertices of those right angles be g, h, i ... f. All these perpendiculars will represent two sets of parallels. The triangles Aal, abg, ... efk will be equal to one another, because they have one side and two adjacent angles equal resp. in all of them, those adjacent angles being alternate angles (Theorem 20, section 2 and Euel. propos. XXVI). Therefore, Al = ag = bh ... ek and therefore, also according to (Euel. propos. XXXIII), Al = lm = mn ... pq. For the same reason a,a = Al; b,b = Am; c,c = An ... f,f = Aq. But Aq containing m+1 times Al is greater than AB; hence f,f > AB; in other words the distance of f from AD is greater than the distance between the parallels BC, AD which is everywhere the same and equal to AB; hence f must lie beyond the space inclosed between both the parallels and, since it is a pt. on AE. AE must intersect BC — pt. f' below.

Q.E.D.

Cor. I Two str. lines in the same pl. which cannot meet however far produced both ways are parallel to one another. For any pt. in one of them is from any pt. in the other at a right angle with that line, wherever they could meet on the side of the right angle it must therefore

it is right or else it & both the lines must be parallel. (Theorem 20 Defin. 11).

Q.E.D.

Cor. II. Any str. line intersecting one of a pair of ... produced sufficiently far must meet the second.

For, being in one pl. with both, were it not to meet the second somewhere, it would be parallel to it (preceding corollary) and hence also to the first (Euc. propos. XXX) which is impossible, therefore, it must intersect the other, somewhere at a finite distance.

Q.E.D.

... of parallels is well ... of several ... is ... established.

We see from, that the ... of the Euclidian ... on a ... arbitrary assumptions, verified by experience to a very ... degree of approximation. ... plotted in the ... great degree, what the ... only ... empirical ... of ... of ... from ...

1) Let the whole original manifold be S, which we suppose to be continuous, i.e. any two different positions in it can be reached from one another only through an unbroken series of other positions, all in S, whose number is infinite.

2) Let an invariable piece of it—U, endowed with impenetrability, rigidity and infinite divisibility be imagined as capable of changing its position as a whole in S. The movability of U is characterized by the fact, that the mutual disposition, or arrangement, and the relation of the parts of U, obtained in a determinate way by any arbitrary subdivision is to remain unaltered, w. r. to each other and w. r. to the whole of U, whereby U is an entire manifold in itself. We say that internal motion of parts of U within U is excluded, and any two parts that have been separated from one another by certain continuous series of other parts, in one position of U within S will remain so in any other position. We thus arrive at the notion of Bulk or Volume. Further, this notion is made more precise by postulating that when such a rigid piece has once occupied a definite portion of vacant or unoccupied S, to the exclusion of any other impenetrable piece, it will always be brought into coincidence with that portion again whenever any point of it, no matter how small will have been brought into coincidence with the corresponding part of S it is occupying originally; any other small portion of U will then also have to occupy its original position within S. If we should measure only bulk or volume, we would get only one dimension. The property of impenetrability of the movable piece however, leads us to distinguish a new category of manifold subordinate to the category S and contained in it, in the following way:

3) Suppose U fixed and another piece V of same nature moving up to it, under the boundary of the latter. It will then be prevented from occupying the same space as U by the impenetrability of the two, exhibited in their boundaries so that a certain kind of motion of V is checked when the latter comes into contact with U. Here we have the notion of the boundary of U separating it from vacant S as from other pieces V of same nature as itself.

4) This first derivative boundary of U, which we now denote by U' is neither a portion of U nor of V that has come into contact with it, nor of the vacant S in which U is posited. It may have portions of its own or not. Examples: the limits of a period of time no matter how great have no parts, since there is no possibility of an invariable piece of time to change its position with respect to other invariable pieces and move up to them to become in contact with their boundaries. In our case, however, U' must have parts of its own, as will become evident from the following considerations:

5) By infinite divisibility of a rigid piece V we may arrive at the notion of a plastic substance, each infinitesimal portion of which occupies a corresponding infinitesimal portion of the manifold S and the whole retaining only the property of impenetrability having lost however rigidity. It will represent in fact a plastic substance, the smallest portions of which are easily capable

of separation and change of position w. r. to one another and w. r. to the whole of their aggregate. A rigid piece U considered immersed in this plastic substance will displace a portion of it equivalent to the bulk of the portion immersed. To a smaller portion of U immersed will correspond a smaller bulk of the plastic material displaced and to a greater portion immersed will correspond a greater bulk displaced. And since this displacement is necessarily effected only by the boundary of the immersed portion of U, and since that portion of the plane which has been exposed by removal of U is distinguished from the other portion which separates the immersed portion of U from the non-immersed and which is indeed the prolongation of the original exposed portion of the boundary of the plastic substance V that is now wholly inside U, both the immersed and the non-immersed portions and can consequently have no effect in the act of displacement considered — it follows that to a greater bulk displaced and hence also to a greater portion of U immersed corresponds a greater portion of the boundary U' which we may call U'_i (immersed) as distinguished from the remaining portion of U' which, as belonging to the exposed portion of U only ~~was exposed and natural~~ we may call (the exposed).

By changing continuously the portion of U immersed from an infinitesimal bulk to the whole bulk of U we arrive at the notion of a continuously increasing boundary U'_i, from an infinitesimal to the whole of U', and a

correspondingly decreasing U_e' from the whole of U' to an infinitesimal of U_e' and then zero.

6) The limit, separating the two portions of U' (i.e. U_i' from U_e') at each and every stage of the process, is evidently a new kind of boundary U'', — and the infinitesimal portion of U', between two very near U''s, corresponding to two very nearly equal bulks of U immersed in the continuous process described above, may be denoted by dU', meaning an infinitesimal of U' and the aggregate of all these $\sum dU_K'$, may be taken to equal the whole of U', belonging to the whole of U just as $U = \sum_K dU_K$. (In the text of the larger work it is shown how we arrive at the notion of a homogeneous U' which may serve as a standard of measurement for different U's). A homogeneous U' may be considered a field of motion for finite portions of U', covered by corresponding portions of V' belonging to some moving rigid V which in the process of motion touches U in the variable portions of U' considered. U' in itself, without the piece U which is bounded by it, has an independent existence only as an abstraction (as for instance force without matter). In fact we can speak of it as moving about either in S or in its own region, only in virtue of corresponding motions of V (or V) to which U' (or V') belongs.

7) Cor. It follows also that U', as a boundary between U and unoccupied adjoining portions of S, or

between U and V, must be considered as having the property of impenetrability. In fact it is conceived of as that which prevents any portion of V, no matter how small, from penetrating into the region occupied by any portion of U, and vice versa. But, as a region of motion for portions of itself, i.e. as a manifold in itself, it cannot possess the property of impenetrability and must in fact, as to the same nature w.r.t. portions of V' or U' moving in the whole, as meant S is w.r.t. impenetrable U or V. This is a reason why the boundary between U' and U'_c can be considered rigid, which may also be considered rigid through the medium of an auxiliary V_1 and portions of whose boundary $-V'_c$, very near $V_c (= U'_c, c \equiv$ covered), ~~but not like that moving~~ will then [illegible] to a third rigid piece W, of same nature as U and V a finite portion of whose boundary W'_c is conceived as covering corresponding finite portions of U'_c (fixed w.r.t. V only) and moving in the manifold U', until a finite portion of W'_c comes into coincidence with a corresponding portion V'_c which thus becomes $V'_{cW} = W'_{cV}$.

8) It is important to observe that when a piece of the original manifold $-V$ comes in contact with another piece of the same kind U, in general only finite portions of their boundaries coincide and thus become $U'_c = V'_c$; the remaining portions of their boundaries combine in forming a new combined boundary S'.

piece $(U+V)$ taken as a whole, this ~~xxx~~ combined boundary being now represented by $(U+V)'-U'V'$. It is, therefore, evident that when W considered in (7) moves up to V, the two portions of its boundary- W'_{CU} and W'_{CV} - become now combined into the boundary separating W from the remained piece $(U+V)$, viz. $(W'_{CU}+W'_{CV})=(U+V)'_{CW}$; and in general, for the same reason as above, there will yet remain a finite portion of $(U+V)'$ - spread which we may denote by $(U+V)'_e$.

71 We see now that U'', originally obtained as the limit separating U'_i from U'_e, when V was considered as the plastic substance in which a portion of U was immersed and identified with the boundary separating $(U'_e\ V'_e)$ from U'_e and also from V'_e, in case V is considered rigid and a portion of its boundary V'_e covering an equal portion of U' - by means of the process considered in 7 becomes browsed up into two portions: one lying in the region of $(U+V)'_e$ exclusively and separating U'_{eW} from V'_{eW}, or each of these from the same corresponding portion of $(U'_e=V'_e)$ and the other lying exclusively in the region of $(U+V)'_{eW}$ $=(W'_{CU}+W'_{CV})$ and separating U'_{eW} from V'_{eW}. Let the first be called U''_{eW} and the second U''_{eW}.

10) By introducing another piece T which we make to play the role of W for the browsing up of U''_{eW} into $U''_{e(W,T)}$ and $U''_{eW\ e}$ and then still

another piece I for the drawing up of $U_{(W,T)}$ into two pieces and so on we see that U'' consists of as many pieces as we please. Moreover by moving W to more up to T so that W_U, W_V are in I is changing its position continuously, and is changing its form and magnitude if need be (in the text of the larger work this is shown to be possible by supposing the first derivative boundary to be homogeneous for simplicity or by allowing sufficient plasticity to W) shall always remain during this process in coincidence with an equivalent variable $(U+V)'_{CW}$: we shall convince ourselves that an infinitesimal $d(W_U, W_V)$ in the neighborhood of U''_{CW} remaining always in coincidence with ~~the moving~~ a variable $d(U+V)'_{CW}$ will displace itself and find a region of motion along the element $d(U+V)'$ taken in the neighborhood of U'' and conceived, in toto, as a locus for the different positions of the moving infinitesimal of a portion of the first derivative boundary. U'' itself, therefore as a whole, will prove to be a locus in gen for $U''_{CW} = \lim d(U+V)'_{CW}$. When W comes up to T it is there checked - and we arrive at the conception U''' separating the region of motion for U''_{CW} from the region where such a motion is impossible. It is, of course, the same boundary as that which separated $U''_{CW,CT}$ from $U''_{(W,T)}$ arrived at at the beginning of the present paragraph.

11) In summing up we see that, while S is a region of motion for pieces like U, V etc., the first derivative boundary U' is the limit of dU and not being a portion of U it has portions of its own, being a region of motion for corresponding finite portions of the boundaries of a movable covering piece V. U'' the limit of dU' is dU' = 0, and not being a part of U' is separating the region of motion in contact for a finite piece W from the region where such a motion is impossible, it still has portions of its own. For although W cannot find upon it a region of motion even for a finite portion of its boundary, it can find upon it a region of motion, or the limit of an infinitesimal portion of its boundary, namely for lim dW (= W''). For, as it was shown, W can be made to move so, that while two distinct but contiguous infinite portions of its boundary move each upon a corresponding portion of U' and V' resp., an infinitesimal of its boundary dW' contiguous to W'', and taken as near W'' as we please, on either side of it will move and remain in ~~coincidence~~ with corresponding variable infinitesimal portions of both U' and V'. In passing as near to the limit as we please we come to the conception of portions of the limit W'' itself moving in the whole of it as in a locus in quo.

The third derivative of U is U''', which is the region of motion for the limit of an indefinitely small portion of the boundary of the first order from the region where motion is impossible even for the limit of an infinitesimal portion of such a boundary. An infinitesimal portion of U'', very near U''', on both sides of it, namely dU'', may be considered an element of U'', so that we have $\Sigma dU'' = U''$.

(2) And now, if we agree that any manifold derived from S in the manner indicated, will have to be capable of giving place to impenetrable substance, or to its boundary which is directly connected with the substance we find that the last derived boundary, U''', which does not, even near its limit, afford a region of motion to an infinitesimal of the first derived boundary of the impenetrable substance, has only position but no dimensions, so that we have three categories of space corresponding to three properties of bodies, rigidity, impenetrability and infinite divisibility, each limiting the preceding and limited by the following. These are S a region of motion for pieces of the impenetrable substance itself U, U' &c and their boundaries both finite and infinitesimal; U' first derivative boundary of impenetrable substance U a region of motion in contact of finite pieces like U or U'; U'' a region of motion for a finite portion of the boundary of a piece of

impenetrable substance and lastly, the second derivative boundary, U'' a region of motion for the limit of an infinitesimal portion of the first derivative boundary dU the boundary of this last region of motion is limiting the region of motion for an infinitesimal portion of the boundary belonging to impenetrable substance must be of zero dimensions but still capable of having definite position as the limit of the primary reducible element of Space. It follows therefore, that such case of the boundaries is capable never to correspond limit of being considered as infinitesimal. Vacuum of the category which is bounded by it the last category in order derived, namely U'', has one dimension the one preceding two dimensions and the original one, U or a piece of vacant S as measured by U, has three dimensions.

13) Instead of introducing W, as done in (9), we can prove that U'' has dimensions also in the following way. We have found in 5 8) that when a rigid V covers a portion U' U'' is the boundary separating the two finite portions $(U'_c\ V'_c)$ and $(U'_e + V'_e)$, each of these being capable of being conceived as a continuous aggregate of dU's or dV's or $\Sigma dU'$ or $\Sigma dV'$ where $\lim (U'_\kappa = U''_\kappa$ &c.)

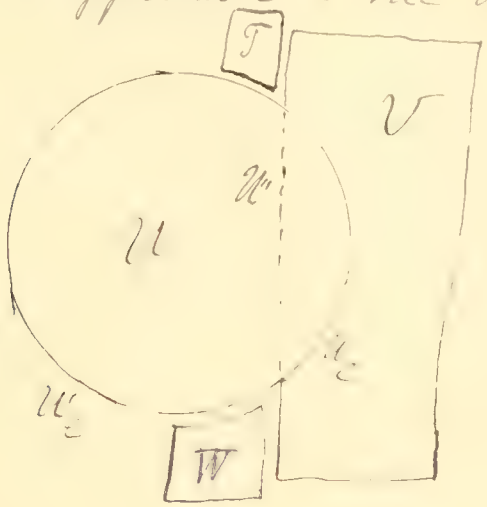
Conceiving now V divided into two portions such that each cover only a finite portion of U'_c (that this can be done is seen from the very process of considering the whole U'_c in the process of immersion

into a plastic V); let one of these be removed. Then, it is evident that the new boundary U''_α will consist only of a portion of the old boundary U'' namely that which belonged to the portion U'_C, covered by the portion of V not removed, U''_2 say, and of a new portion U''_β, taking the place of U''_2 the portion of U'' which belonged to the portion of U'_C that became exposed by the removal of the portion of V. This new portion U''_β, indeed separates U'_{C_2} from U'_C belonging respectively to V_2 and V_1, the portion of V not removed and the portion removed. By changing V_2 in such a way as to keep with it U'_{C_2}, as for instance by removing a new portion from V_2 such as to expose a new finite portion of U'_{C_2}, we shall have to replace it by some new portion U''_γ lying wholly in the region of U'_{C_2} either a portion of U''_β only, or the whole of U''_β or a portion of U''_β and a portion of U''_2 or lastly the whole of U''_2 alone or only a portion of it. We see, however, that U''_2 can be again broken up into two distinct portions, and each of these can again be broken up into two portions, and so on. In this way we can convince ourselves that $U'' = U''_1 + U''_2 + U''_3 + \ldots + U''_n + \ldots$ an infinite series of terms. There is no difficulty in making this series a continuous aggregate of infinitely small elements $\Sigma dU''$ as soon as the existence of a homogeneous U'' is established which could serve as a standard of measurement for all other U''s by a suitable comparison of their infinitesimal elements with

those of the standard superposition in which the boundaries of the elements compared (i.e. U'' & U''') are made to coincide for the two series. The proof, however, that U''' has no dimensions must again be derived from the fact that U, as a whole in itself, is $\lim dU'$, i.e. the limit of a locus ~~of~~ a locus of motion of an infinitesimal element of the first derivative boundary of impenetrable substance, i.e. hence of one dimension only the boundary, therefore of the limit of this locus will have only position and no dimensions as above.

14) Conclusion. By the very process of the deduction of the three subjectively different categories of space (regions of different kinds of motion which, however are the boundaries of one another in a series), we have arrived at the notion of a manifold of two dimensions objectively not independent from the main category but nonetheless having a true abstract reality and which by its very nature, as a manifold in itself, and not as a boundary, is devoid of the property of impenetrability (See No. 7) In this derived manifold therefore boundaries of portions can be established only by means of a piece of the higher manifold, having the property of impenetrability in the region of the dimensions of the derived manifold considered, and for this very reason suggesting a dimension over and above those of the derived manifold. There is no wonder, therefore, that the

general reasoning, applicable to the original manifold, is not applicable to the derived manifold. The figure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